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GROWTH AND DECAY OF ACCELERATION WAVES IN NON-ASSOCIATIVE ELASTIC–PLASTIC FLUID-SATURATED POROUS MEDIA

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Abstract—The modes under which acceleration waves can propagate in elastic–plastic fluid-saturated porous media have been obtained in Loret and Harireche [*J. Mech. Phys. Solids* **39**, 569–606 (1991)]. The implications of the existence of complex wave-speeds are analyzed here. First, like elastic mixtures, elastic–plastic mixtures with an associative flow rule may be characterized by a positive and finite decay coefficient: acceleration waves propagate with an amplitude that either strictly decreases or remains constant. Moreover, the range of variation of the decay coefficient is the same as for elastic mixtures. In contrast, when the squares of the wave-speeds are real, non-associative flow rules may give rise to negative and/or unbounded decay coefficients. When the squares of the wave-speeds are complex, the so-called flutter phenomenon, the decay coefficient is found to be positive and finite.

The analytical derivations require the material state on the wave front to be constant; on the other hand, the analysis is valid independently of the compressibilities of the solid and fluid constituents. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Instability of the flow of fluid-saturated porous media has been analyzed for simple boundary value problems in both quasi-static and dynamic contexts (e.g. Rice, 1975; Vardoulakis, 1986). In view of three-dimensional numerical applications, the analysis to be presented in this note falls in the framework of the theory of elastic–plastic mixtures obeying deviatoric associativity. For such a behaviour, finite element simulations of strain-localization are described in Loret and Prevost (1991). The phenomenon displays a typical velocity pattern where the fluid is attracted to the shear-bands which are zones of intense shearing and dilation. It is known that strain-localization can be viewed as the limit of an acceleration wave whose speed vanishes, hence the term stationary discontinuity (Hill, 1962). An analysis of the nature of the wave-speeds in elastic–plastic mixtures is performed in Loret and Harireche (1991); conditions for the existence of strain-localization are derived and, in addition, it is shown that non-associativity may imply the existence of complex squares of wave-speeds, the so-called flutter phenomenon. For hypoelastic materials, which have no unloading branch, the interpretation of flutter given by Rice (1976) is to think of harmonic waves that propagate through the material but whose amplitude increases.

Here we consider the propagation of acceleration waves. Under simplifying assumptions concerning the material state on the wave-front, we find that, like elastic mixtures (Biot, 1956; Bowen 1976), elastic–plastic mixtures with an associative flow rule may be characterized by a positive and finite decay coefficient, that is, acceleration waves propagate with an amplitude that either strictly decreases or remains constant. In contrast, outside the flutter region, non-associative flow rules may give rise to negative and unbounded decay coefficients for some material parameters and certain propagation directions. On the other hand, inside the flutter region, the decay coefficient is found to be positive and finite.

The paper is organized as follows. First the constitutive equations for an elastic–plastic mixture developed in Loret and Harireche (1991) are briefly recalled (Section 2) and the eigenvalue problems yielding the wave-speeds and modes under which acceleration waves can propagate are deduced (Section 3). Next, the evolution of the amplitude of an acceleration wave is given an explicit form characterized by a decay coefficient (Section 4). For both compressible and incompressible fluid and solid constituents, a qualitative and quantitative analysis of the sign and boundedness of the decay coefficient is presented and the implications of the existence of real and complex wave-speeds are emphasized (Sections 5 and 6).

The analysis presented in this note is restricted to small perturbations relative to a natural configuration. Unless stated explicitly, we use the convention of summation over repeated mute indices.

2. CONSTITUTIVE EQUATIONS FOR A MIXTURE

We consider continuum media that are linear isotropic with respect to their elastic properties but whose plastic properties may embody any kind of anisotropy. The constitutive equations relate the rates of the partial stress-tensors $\dot{\mathbf{t}}^\alpha$ to the velocity-gradients $\partial \mathbf{v}_\alpha/\partial \mathbf{x}$; here and throughout the note, the greek indices α and β apply to the solid ($\alpha = \beta = s$) and fluid ($\alpha = \beta = w$) phases or constituents.

Each phase α is endowed with its own velocity $\mathbf{v}_\alpha$. Both solid and fluid phases are inviscid, both solid and fluid constituents are *a priori* compressible. The fluid is a perfect fluid whose partial pressure is denoted by p^w; thus $\mathbf{t}^w = -p^w\boldsymbol{\delta}$, with $\boldsymbol{\delta}$ denoting the identity tensor in a three-dimensional space.

The rate constitutive equations developed in Loret and Harireche (1991) can be recast in the following form

$$\dot{\mathbf{t}}^s = \mathscr{A}^{ss} : \frac{\partial \mathbf{v}_s}{\partial \mathbf{x}} + \mathscr{A}^{sw} : \frac{\partial \mathbf{v}_w}{\partial \mathbf{x}} \tag{1}$$

$$\dot{\mathbf{t}}^w = \mathscr{A}^{ws} : \frac{\partial \mathbf{v}_s}{\partial \mathbf{x}} + \mathscr{A}^{ww} : \frac{\partial \mathbf{v}_w}{\partial \mathbf{x}}, \tag{2}$$

where the fourth-order tensors $\mathscr{A}^{\alpha\beta}$, which are endowed with the minor symmetries in their first two and last two indices, are rank-one modifications with respect to the elastic constitutive contributions:

$$\mathscr{A}^{ss} = \mathbf{E}^s - \frac{l}{H}(\mathbf{E}'^s : \mathbf{P}) \otimes (\mathbf{Q} : \mathbf{E}'^s), \quad \mathscr{A}^{sw} = \lambda_{sw}\mathbf{I} - \frac{l}{H}(\mathbf{E}'^s : \mathbf{P}) \otimes (\mathbf{Q} : \lambda'_{sw}\mathbf{I}) \tag{3}$$

$$\mathscr{A}^{ws} = \lambda_{sw}\mathbf{I} - \frac{l}{H}(\lambda'_{sw}\mathbf{I} : \mathbf{P}) \otimes (\mathbf{Q} : \mathbf{E}'^s), \quad \mathscr{A}^{ww} = \lambda_w\mathbf{I} - \frac{l}{H}(\lambda'_{sw}\mathbf{I} : \mathbf{P}) \otimes (\mathbf{Q} : \lambda'_{sw}\mathbf{I}). \tag{4}$$

In the above formulas, $\otimes$ denotes a dyadic product, a dot '$\cdot$' denotes a scalar product, a double-dot ':' stands for the trace of the scalar product, e.g. in cartesian axes $\mathbf{P} : \mathbf{Q} = P_{ij}Q_{ij}$, $(\mathbf{E}'^s : \mathbf{P})_{ij} = E'^s_{ijkl}P_{kl}$ and $\mathbf{P}$ and $\mathbf{Q}$ are the unit outward normals to the plastic potential and yield surface, respectively, both being assumed to be smooth. l is a loading/unloading index, namely $l = 1$ for plastic loading, when the stress point is on the yield surface and the plastic index λ is strictly positive:

$$\lambda = \frac{1}{H}\mathbf{Q}:\left(\mathbf{E}'^{s}:\frac{\partial \mathbf{v}_s}{\partial \mathbf{x}} + \lambda'_{sw}\mathbf{I}:\frac{\partial \mathbf{v}_w}{\partial \mathbf{x}}\right) > 0 \tag{5}$$

and $l = 0$ otherwise. The modulus H is assumed to be strictly positive in order to exclude a locking behaviour:

$$H = h + h_e > 0, \quad h_e = \mathbf{Q}:\mathbf{E}''^{s}:\mathbf{P}; \tag{6}$$

$\mathbf{E}^s$, $\mathbf{E}'^s$ and $\mathbf{E}''^s$ are isotropic fourth-order tensors. In the basis $(\mathbf{I}, \mathbf{J})$, they are defined by their respective components $(\lambda_s, 2\mu_s)$, $(\lambda'_s, 2\mu_s)$ and $(\lambda''_s, 2\mu_s)$:

$$\lambda'_s = \lambda_s - \frac{n^s}{n^w}\lambda_{sw}, \quad \lambda''_s = \lambda'_s - \frac{n^s}{n^w}\lambda'_{sw} \quad \text{and} \quad \lambda'_{sw} = \lambda_{sw} - \frac{n^s}{n^w}\lambda_w; \tag{7}$$

$n^\alpha \in]0, 1[$, $\alpha = s, w$ is the volume fraction of phase α:

$$n^s + n^w = 1; \tag{8}$$

$\mathbf{I}$ and $\mathbf{J}$ are fourth-order basis tensors defined in cartesian axes as follows:

$$I_{ijkl} = \delta_{ij}\delta_{kl}, \quad J_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{9}$$

The elastic material response is defined by the four constants λ_s, λ_{sw}, λ_w and μ_s which can be related to measurable quantities like the Biot and Skempton parameters (see e.g. Bowen, 1982; Loret and Harireche, 1991). The requirement that the elastic strain energy function be positive-definite places restrictions on the ranges of these parameters, namely

$$\mu_s > 0, \quad \lambda_w > 0, \quad \lambda_s + \frac{2}{3}\mu_s > 0 \quad \text{where } \lambda_s = \lambda_s - \frac{\lambda_{sw}^2}{\lambda_w}. \tag{10}$$

These restrictions are known to imply the existence of real and strictly positive elastic wave-speeds.

Remark 2.1

Symmetry of the constitutive equations. Observe that, if the solid phase, or skeleton, obeys an associative flow rule, that is $\mathbf{P} = \mathbf{Q}$, the constitutive equations of the porous medium display the major symmetry property.

Remark 2.2

Stress-rates. As for the rate of stress involved in the definition of the constitutive equations, we shall neglect the corotational terms, so that the superimposed dot denotes *a priori* a material time-derivative which, in the following linearized analysis, we approximate by the partial derivative with respect to time $\partial/\partial t$.

Remark 2.3

Sands and incompressible constituents. The above description may be applied to describe the behaviour of rocks and sands. For rocks, the compressibilities of the solid and fluid constituents do not usually differ by many orders of magnitude. On the other hand, the grains in sands are much less compressible than the fluid so that the elastic parameters satisfy the relation (Bowen, 1982):

$$\frac{\lambda_{sw}}{\lambda_w} = \frac{n^s}{n^w} \quad \text{whence } \lambda_{sw} = 0 \quad \text{and} \quad \lambda_s = \lambda_s = \lambda_s. \tag{11}$$

If both constituents are incompressible and in absence of chemical reactions, the conservation of mass of both constituents constraints the volumetric strain-rates:

$$n^s \operatorname{div} \mathbf{v}_s + n^w \operatorname{div} \mathbf{v}_w = 0. \tag{12}$$

Moreover, for the fluid-pressure to be indeterminate, the following asymptotic relations must hold:

$$\lambda_w \to \infty, \quad \lambda_{sw} \to \infty, \quad \text{while } \lambda_{sw} = 0 \quad \text{and} \quad \lambda_s = \lambda_s = \lambda_s < \infty. \tag{13}$$

To deduce the constitutive equations of the medium with both constituents incompressible, one can use the following limit procedure:

$$\mathscr{I} \begin{cases} \text{replace } \lambda_s \text{ by } \lambda_s + \lambda_{sw}^2/\lambda_w, \\ \text{use eqn (11) and take the limits (13).} \end{cases} \tag{14}$$

The result is best expressed in terms of Terzaghi stress $\mathbf{t}'^s$:

$$\mathbf{t}^s = \mathbf{t}'^s + \frac{n^s}{n^w}\mathbf{t}^w, \tag{15}$$

and then

$$\dot{\mathbf{t}}'^s = \mathscr{A}^{*ss} : \frac{\partial \mathbf{v}_s}{\partial \mathbf{x}}, \quad \mathscr{A}^{*ss} = \mathbf{E}^{*s} - \frac{l}{H}(\mathbf{E}^{*s} : \mathbf{P}) \otimes (\mathbf{Q} : \mathbf{E}^{*s}), \tag{16}$$

where l is another loading/unloading index, namely $l = 1$ for plastic loading, when the stress point is on the yield surface and the plastic index λ is strictly positive:

$$\lambda = \frac{1}{H}\left(\mathbf{Q} : \mathbf{E}^{*s} : \frac{\partial \mathbf{v}_s}{\partial \mathbf{x}}\right) > 0, \quad H = h + h_e > 0, \quad h_e = \mathbf{Q} : \mathbf{E}^{*s} : \mathbf{P} \tag{17}$$

and $l = 0$ otherwise. The elastic tensor $\mathbf{E}^{*s}$ has components $(\lambda_s, 2\mu_s)$ in the basis $(\mathbf{I}, \mathbf{J})$. For sands and incompressible constituents, the constitutive equations above reduce to those of Prevost (1980).

3. ACCELERATION WAVES IN THE ELASTIC-PLASTIC MIXTURE

For each phase, $\alpha = s, w$, of the porous medium, the balance of momentum

$$\operatorname{div} \mathbf{t}^\alpha + \hat{\mathbf{p}}_\alpha + \rho^\alpha(\mathbf{b}_\alpha - \mathbf{a}_\alpha) = 0 \quad \text{(no sum over } \alpha) \tag{18}$$

involves, in addition to the usual terms present in single phase solids, namely divergence of the stress tensor, body force per unit mass $\mathbf{b}_\alpha$ and acceleration $\mathbf{a}_\alpha$, the apparent mass density ρ^α and the momentum supply $\hat{\mathbf{p}}_\alpha$ by the rest of the mixture; momentum supplies are subject to the constraint

$$\hat{\mathbf{p}}_s + \hat{\mathbf{p}}_w = 0. \tag{19}$$

The balance of momentum equation holds pointwise, except along a singular surface of order 1 (see e.g. Eringen and Ingram, 1965). Singular surfaces of order 1 display a velocity discontinuity, while, across singular surfaces of order 2, the velocity is continuous but the acceleration and the velocity gradient are not. Here, only the existence and propagation of the latter are considered.

Let $\phi(\mathbf{x}, t)$ be a vector- or tensor-valued function which is pointwise continuous and which has continuous derivatives except on a singular surface Σ of local normal $\mathbf{n}(\mathbf{x}, t)$ whose speed of displacement will be denoted W; let $\boldsymbol{\eta}(\mathbf{x}, t)$ be the normal jump of the spatial gradient of ϕ. The jumps across Σ of the spatial- and time-derivatives of ϕ are connected by Hadamard's compatibility conditions (Truesdell and Toupin, 1960, eqn 180.5):

$$\left[\!\left[\frac{\partial \phi}{\partial \mathbf{x}}\right]\!\right] = \boldsymbol{\eta} \otimes \mathbf{n}, \quad \left[\!\left[\frac{\partial \phi}{\partial t}\right]\!\right] = -W\boldsymbol{\eta}. \tag{20}$$

Since we envisage a linearized analysis around an equilibrium state, the mass density ρ^α in eqn (18) is fixed to its reference value and the acceleration $\mathbf{a}_\alpha$ is approximated by $\partial \mathbf{v}_\alpha / \partial t$; thus we shall make no difference between the speed of displacement W and the two speeds of propagation with respect to the particles of each phase $c_\alpha = W - \mathbf{v}_\alpha \cdot \mathbf{n}$.

To compute the acceleration wave-speeds, we need a constitutive assumption for the momentum supplies. They will be assumed continuous across singular surfaces of order 2. Therefore application of Hadamard's compatibility conditions with $\phi = \mathbf{t}^\alpha$ and $\phi = \mathbf{v}^\alpha$ to the balance of momentum equations of the two phases yields

$$\left[\!\left[\frac{\partial \mathbf{t}^\alpha}{\partial t}\right]\!\right] \cdot \mathbf{n} = (\rho^\alpha W^2)\boldsymbol{\eta}_\alpha, \quad \text{(no sum over } \alpha). \tag{21}$$

Insertion of the constitutive equations (1) and (2) in the above relations, accounting for Remark 2.2., yields a generalized eigenvalue problem for the squares of the wave-speeds W^2:

$$(\mathbf{A} - W^2\mathbf{M}) \cdot \boldsymbol{\eta} = 0 \quad \text{or} \quad \begin{bmatrix} \mathbf{A}^{ss} - W^2\mathbf{M}^{ss} & \mathbf{A}^{sw} \cdot \mathbf{n} \\ (\mathbf{n} \cdot \mathbf{A}^{ws})^{\mathrm{T}} & \mathbf{n} \cdot (\mathbf{A}^{ww} - W^2\mathbf{M}^{ww}) \cdot \mathbf{n} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_s \\ \boldsymbol{\eta}_w \cdot \mathbf{n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22}$$

where

$$\mathbf{A}^{\alpha\beta} = \mathbf{n} \cdot \mathscr{A}^{\alpha\beta} \cdot \mathbf{n}, \quad \mathbf{M}^{\alpha\alpha} = \rho^\alpha \boldsymbol{\delta} \quad \text{(no sum over } \alpha), \quad \text{for } \alpha, \beta = s, w. \tag{23}$$

To derive eqn (22), one has accounted for the fact that the fluid is perfect, thus the discontinuity of the spatial gradient of the velocity in the fluid $\boldsymbol{\eta}_w$ is aligned with the wave normal $\mathbf{n}$ due to eqn (21)

$$\boldsymbol{\eta}_w = (\boldsymbol{\eta}_w \cdot \mathbf{n})\mathbf{n}. \tag{24}$$

Also, the special structure of the matrices $\mathbf{A}^{\alpha\beta}$ has been exploited (see Loret and Harireche, 1991, eqns 4.4–7):

$$\mathbf{A}^{sw} = (\mathbf{A}^{sw} \cdot \mathbf{n}) \otimes \mathbf{n}, \quad \mathbf{A}^{ws} = \mathbf{n} \otimes (\mathbf{n} \cdot \mathbf{A}^{ws}), \quad \mathbf{A}^{ww} = (\mathbf{n} \cdot \mathbf{A}^{ww} \cdot \mathbf{n})\mathbf{n} \otimes \mathbf{n}. \tag{25}$$

In addition, one has assumed the two sides of the wave-front to follow the plastic branch of the constitutive one. Consequently, there are at maximum four possible modes of propagation, each one associated with a solution W^2 of eqn (22). Loret and Harireche (1991) present a detailed analysis of the nature, real or complex, of the wave-speeds; they also order the wave-speeds when they are real.

Remark 3.1

Reduction to an eigenvalue problem. By applying the following generic transformations to eigenvectors and to matrices, respectively,

$$\mathbf{V} \rightarrow \mathbf{V}^{\#} = \mathbf{M}^{1/2} \cdot \mathbf{V}, \quad \mathbf{T} \rightarrow \mathbf{T}^{\#} = \mathbf{M}^{-1/2} \cdot \mathbf{T} \cdot \mathbf{M}^{-1/2} \tag{26}$$

the generalized eigenvalue problem (22) may be reduced to an eigenvalue problem:

$$(\mathbf{A}^{\#} - W^2 \mathbf{I}_4) \cdot \boldsymbol{\eta}^{\#} = 0, \quad \begin{bmatrix} \mathbf{A}^{\#ss} - W^2 \boldsymbol{\delta} & \mathbf{A}^{\#sw} \cdot \mathbf{n} \\ (\mathbf{n} \cdot \mathbf{A}^{\#ws})^{\mathrm{T}} & \mathbf{n} \cdot (\mathbf{A}^{\#ww} - W^2 \boldsymbol{\delta}) \cdot \mathbf{n} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_s^{\#} \\ \boldsymbol{\eta}_w^{\#} \cdot \mathbf{n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{27}$$

where $\mathbf{I}_4$ is the identity matrix in $\mathscr{R}^4$ and

$$\mathbf{A}^{\#\alpha\beta} = \frac{\mathbf{n} \cdot \mathscr{A}^{\alpha\beta} \cdot \mathbf{n}}{\sqrt{\rho^{\alpha} \rho^{\beta}}}, \quad \text{for } \alpha, \beta = s, w \text{ (no sum over } \alpha, \beta). \tag{28}$$

Equations (27) and (28) show clearly that the major symmetry of the constitutive equations is inherited by the normalized pseudo-acoustic matrix $\mathbf{A}^{\#}$. Thus, for an elastic–plastic skeleton obeying an associative flow rule, the eigenvalue problem is symmetric and, therefore, the squares of wave-speeds and eigenvectors are real.

Remark 3.2

Incompressible constituents. If both constituents are incompressible, the discontinuities $\boldsymbol{\eta}_{\alpha}$ are constrained by the incompressibility condition (12), namely:

$$n^s(\boldsymbol{\eta}_s \cdot \mathbf{n}) + n^w(\boldsymbol{\eta}_w \cdot \mathbf{n}) = 0. \tag{29}$$

Therefore the number of modes of propagation is now at maximum three. The associated eigenvalue problem can be obtained using the limit procedure ($\mathscr{I}$), or directly from the balances of momentum by eliminating the fluid pressure (Loret and Harireche, 1991)

$$(\mathbf{A}^{*ss} - W^2 \mathbf{M}) \cdot \boldsymbol{\eta}_s = 0 \tag{30}$$

where

$$\mathbf{A}^{*ss} = \mathbf{n} \cdot \mathscr{A}^{*ss} \cdot \mathbf{n}, \quad \mathbf{M} = \rho^s(\boldsymbol{\delta} + (r-1)\mathbf{n} \otimes \mathbf{n}) \tag{31}$$

and r is a scalar greater than 1:

$$r = 1 + \left(\frac{n^s}{n^w}\right)^2 \frac{\rho^w}{\rho^s}. \tag{32}$$

Therefore, if necessary, reduction to an eigenvalue problem is still possible and

$$\mathbf{M}^{1/2} = \sqrt{\rho^s}(\boldsymbol{\delta} + (\sqrt{r} - 1)\mathbf{n} \otimes \mathbf{n}). \tag{33}$$

4. PROPAGATION OF PLANE ACCELERATION WAVES

From the previous section, we know the modes under which plane acceleration waves propagate. We shall now study how their amplitudes vary in time. This analysis requires knowledge of the evolution of the wave-speed on the wave front. In contrast to linear elasticity where the wave-speeds in a given direction of propagation depend on material constants only, the elastic–plastic wave-speeds depend on the material state, that is stresses and hardening parameters. In a general initial- and boundary-value problem, the latter are functions of point. To be able to carry the analysis further, we consider a hypothetical situation where the material state on the wave-front is constant; then, so remains the speed of displacement W.

To proceed further, we need also to substantiate the qualitative assumption made in the previous section concerning the momentum supplies. Indeed, we shall adopt simply an isotropic Darcy's law that introduces a single constant material parameter $\xi > 0$ proportional to the inverse of the permeability (Bowen, 1976):

$$\hat{\mathbf{p}}_s = -\hat{\mathbf{p}}_w = -\xi(\mathbf{v}_s - \mathbf{v}_w). \tag{34}$$

Notice that buoyancy terms need not to be included since they would disappear anyway in this linearized analysis. Moreover, the rates of body forces $\mathbf{b}_\alpha$, $\alpha = s, w$, are assumed to be continuous across the wave-front. Thus upon derivation with respect to time, the linearized equations of balance of momentum (18) yield the discontinuity equations:

$$\rho^\alpha\left[\!\left[\frac{\partial^2 \mathbf{v}_\alpha}{\partial t^2}\right]\!\right] = \left[\!\left[\operatorname{div}\frac{\partial \mathbf{t}^\alpha}{\partial t}\right]\!\right] - \varepsilon_\alpha \xi\left(\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right] - \left[\!\left[\frac{\partial \mathbf{v}_w}{\partial t}\right]\!\right]\right) \tag{35}$$

with $\varepsilon_\alpha = 1$ for $\alpha = s$, $\varepsilon_\alpha = -1$ for $\alpha = w$. The evolution of the amplitude of the acceleration wave-front is expressed through a differential equation in terms of the *displacement derivative* (Truesdell and Toupin, 1960, eqn 179.8):

$$\frac{\delta}{\delta t}\left[\!\left[\frac{\partial \mathbf{v}_\alpha}{\partial t}\right]\!\right] = \frac{\partial}{\partial t}\left[\!\left[\frac{\partial \mathbf{v}_\alpha}{\partial t}\right]\!\right] + \frac{\partial}{\partial \mathbf{x}}\left[\!\left[\frac{\partial \mathbf{v}_\alpha}{\partial t}\right]\!\right] \cdot (W\mathbf{n}). \tag{36}$$

Now, since the velocity is continuous across the front that travels at constant speed W in a given direction $\mathbf{n}$, the *iterated kinematical compatibility equation* (Truesdell and Toupin, 1960, eqn 181.8) simplifies to

$$2\frac{\delta}{\delta t}\left[\!\left[\frac{\partial \mathbf{v}_\alpha}{\partial t}\right]\!\right] = \left[\!\left[\frac{\partial^2 \mathbf{v}_\alpha}{\partial t^2}\right]\!\right] - W^2 \mathbf{\Delta}_\alpha \tag{37}$$

where the vector $\mathbf{\Delta}_\alpha$ is called *induced discontinuity*:

$$\mathbf{\Delta}_\alpha = \mathbf{n} \cdot \left[\!\left[\frac{\partial^2 \mathbf{v}_\alpha}{\partial \mathbf{x}^2}\right]\!\right] \cdot \mathbf{n}, \quad \Delta_{\alpha k} = \left[\!\left[\frac{\partial^2 v_{\alpha k}}{\partial x_i \partial x_j}\right]\!\right] n_i n_j, \quad \text{for } i, j, k \in [1, 3]. \tag{38}$$

Due to the *iterated geometrical condition of compatibility* (Truesdell and Toupin, 1960, eqn 176.8), the discontinuity of the second spatial derivative of the velocity can be cast into a dyadic form that also involves the induced discontinuity:

$$\left[\!\left[\frac{\partial^2 \mathbf{v}_\alpha}{\partial \mathbf{x}^2}\right]\!\right] = \mathbf{\Delta}_\alpha \otimes \mathbf{n} \otimes \mathbf{n}, \quad \left[\!\left[\frac{\partial^2 v_{\alpha k}}{\partial x_i \partial x_j}\right]\!\right] = \Delta_{\alpha k} n_i n_j, \quad \text{for } i, j, k \in [1, 3]. \tag{39}$$

When introduced in the rate form of the balance of momentum (35) and in the constitutive equations (1) and (2), the above iterated compatibility equations yield the evolution equations for the discontinuities of acceleration $[\![\partial \mathbf{v}_\alpha / \partial t]\!]$, $\alpha = s, w$:

$$2\rho^s \frac{\delta}{\delta t}\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right] = (\mathbf{A}^{s\beta} - W^2 \rho^\beta \delta_{s\beta}\boldsymbol{\delta}) \cdot \boldsymbol{\Delta}_\beta - \xi\left(\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right] - \left[\!\left[\frac{\partial \mathbf{v}_w}{\partial t}\right]\!\right]\right), \tag{40}$$

$$2\rho^w \frac{\delta}{\delta t}\left[\!\left[\frac{\partial \mathbf{v}_w}{\partial t}\right]\!\right] = (\mathbf{A}^{w\beta} - W^2 \rho^\beta \delta_{w\beta}\boldsymbol{\delta}) \cdot \boldsymbol{\Delta}_\beta + \xi\left(\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right] - \left[\!\left[\frac{\partial \mathbf{v}_w}{\partial t}\right]\!\right]\right). \tag{41}$$

Due to the particular structure of the matrices $\mathbf{A}^{sw}$ and $\mathbf{A}^{ww}$, eqn (25), the products $\mathbf{A}^{sw} \cdot \boldsymbol{\Delta}_w$ and $\mathbf{A}^{ww} \cdot \boldsymbol{\Delta}_w$ involve only the normal component $\boldsymbol{\Delta}_w \cdot \mathbf{n}$. Since the acceleration in the fluid is aligned with the propagation direction $\mathbf{n}$, eqn (24), the wave front is characterized by the four component vector $\mathbf{V} = [[\![\partial \mathbf{v}_s/\partial t]\!]\ [\![\partial \mathbf{v}_w/\partial t]\!] \cdot \mathbf{n}]^T$. Therefore the evolution of the wave-front is given by eqn (40) and the normal component of eqn (41):

$$2\mathbf{M} \cdot \frac{\delta}{\delta t}\mathbf{V} = (\mathbf{A} - W^2 \mathbf{M}) \cdot \boldsymbol{\Delta} - \xi \frac{\mathbf{E}}{\rho^s} \cdot \mathbf{V} \tag{42}$$

with

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Delta}_s \\ \boldsymbol{\Delta}_w \cdot \mathbf{n} \end{bmatrix} \quad \text{and} \quad \mathbf{E} = \rho^s \begin{bmatrix} \boldsymbol{\delta} & -\mathbf{n} \\ -\mathbf{n}^T & 1 \end{bmatrix}. \tag{43}$$

Since we assume the state on the wave-front to be fixed, the unit left and right eigenvectors of the generalized eigenvalue problem (22), namely $\mathbf{e}^L$ and $\mathbf{e}^R$, are constant. Therefore, multiplication of the evolution equation (42) by the unit left eigenvector $\mathbf{e}^L$ yields a differential equation that is easily integrated to yield the amplitude $|\mathbf{V}| = (|[\![\partial \mathbf{v}_s/\partial t]\!]|^2 + |[\![\partial \mathbf{v}_w/\partial t]\!] \cdot \mathbf{n}|^2)^{1/2}$:

$$\frac{|\mathbf{V}|(t)}{|\mathbf{V}|(0)} = \left|\exp\left(-\frac{\xi}{2\rho^s} X t\right)\right|, \quad X = \frac{\mathbf{e}^L \cdot \mathbf{E} \cdot \mathbf{e}^R}{\mathbf{e}^L \cdot \mathbf{M} \cdot \mathbf{e}^R}. \tag{44}$$

Since the coefficient ξ is a positive real number, growth or decay of the wave-front is governed by the sign of the non-dimensional decay coefficient X when the latter is a real number. For an elastic–plastic skeleton obeying an associative flow rule, the eigenvalue problem, eqn (22), is symmetric (Remark 3.1), thus the wave-speeds are real and the left and right eigenvectors are one and the same. Since the matrix $\mathbf{M}$, eqn (23), is definite-positive and the matrix $\mathbf{E}$, eqn (43), is symmetric semi-definite positive, the coefficient X is then positive or zero and propagation is accompanied by a wave-front whose amplitude either strictly decays or remains constant.

This conclusion does not hold *a priori* if the solid skeleton obeys a non-associative flow rule, $\mathbf{P} \neq \mathbf{Q}$, in eqns (3) and (4). Moreover, since then the wave-speeds may be complex, the decay coefficient is the real part of X, $\mathcal{Re}(X)$, rather than X itself.

The evolution equation (42) yields not only the decay coefficient but also the part of the induced discontinuity which is orthogonal to the eigenmode $\mathbf{e}^R$; in fact, when the wave-speed has single multiplicity, one can show that this part is uniquely defined. The situation is more complex for multiple wave-speeds where each case needs to be treated separately (see case d, Section 5, concerning incompressible constituents).

Remark 4.1

On a direct evaluation of the decay coefficient X. To obtain the decay coefficient for a given wave-speed W, the natural approach is to calculate first the associated left and right eigenvectors $\mathbf{e}^R$ and $\mathbf{e}^L$. However, if W is a wave-speed of single multiplicity, considerable algebraic simplifications result from the alternative procedure below.

Let $W^2(0) = W^2$ be an eigenvalue with single multiplicity of the generalized eigenproblem (22) and let $\mathbf{e}^R(0) = \mathbf{e}^R$ and $\mathbf{e}^L(0) = \mathbf{e}^L$ be the associated right and left eigenvectors. Consider now the extended eigenvalue problem:

$$(\mathbf{A}+\zeta\mathbf{E}-W^2(\zeta)\mathbf{M})\cdot\mathbf{e}^R(\zeta) = 0. \tag{45}$$

For ζ a sufficiently small real number, the eigenvalue $W^2(\zeta)$ is an analytic function of ζ (Stoer and Bulirsch, 1980, p. 389). Therefore scalar multiplication of eqn (45) by the left eigenvector $\mathbf{e}^L(0) = \mathbf{e}^L$, differentiation with respect to ζ of the resulting product and estimation of the derivative at $\zeta = 0$ yields:

$$\frac{\mathrm{d}W^2}{\mathrm{d}\zeta}(\zeta = 0) = X = \frac{\mathbf{e}^L\cdot\mathbf{E}\cdot\mathbf{e}^R}{\mathbf{e}^L\cdot\mathbf{M}\cdot\mathbf{e}^R}. \tag{46}$$

Notice that the products by $\mathbf{e}^L$ are not inner products in the usual sense, that is, they remain unchanged even if $\mathbf{e}^L$ is complex (Wilkinson, 1965, eqn (3.10), p. 4).

The procedure breaks down at an eigenvalue of multiplicity greater than one, where the nature of the eigenspace may be more complex. Let us simply consider the case of a double eigenvalue, for **A** a non-symmetric matrix. Then, the eigenvectors may not be unique or, if the eigenspace is defective, the left and right eigenvectors may be well-defined and then they are orthogonal with respect to the matrix **M**, that is (Wilkinson, 1965, eqn (7.5), p. 10):

$$\mathbf{e}^L\cdot\mathbf{M}\cdot\mathbf{e}^R = 0 \tag{47}$$

which is in strong contrast with the case of single multiplicity, where this equality occurs only for $\mathbf{e}^L$ and $\mathbf{e}^R$ associated to distinct eigenvalues. This result suggests that, in the vicinity of a wave-speed of multiplicity greater than 1, the decay coefficient for non-associative flow rules may become unbounded: this phenomenon will actually be observed for double wave-speeds different from elastic wave-speeds.

Remark 4.2

The case of incompressible constituents. For incompressible constituents, the decay coefficient X can be obtained by considering first the compressible case above and then using the limit procedure ($\mathscr{I}$). Alternatively, one can proceed directly using the fact that the fluid pressure is indeterminate. Using Terzaghi's effective stress, eqn (15), equations (35) become:

$$2\rho^s\frac{\delta}{\delta t}\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right] = (\mathbf{A}^{*ss}-\rho^s W^2\boldsymbol{\delta})\cdot\boldsymbol{\Delta}_s+\frac{n^s}{n^w}\left[\!\left[\operatorname{div}\frac{\partial \mathbf{t}^w}{\partial t}\right]\!\right]-\xi\left(\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right]-\left[\!\left[\frac{\partial \mathbf{v}_w}{\partial t}\right]\!\right]\right) \tag{48}$$

$$2\rho^w\frac{\delta}{\delta t}\left[\!\left[\frac{\partial \mathbf{v}_w}{\partial t}\right]\!\right] = \left[\!\left[\operatorname{div}\frac{\partial \mathbf{t}^w}{\partial t}\right]\!\right]-\rho^w W^2\boldsymbol{\Delta}_w+\xi\left(\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right]-\left[\!\left[\frac{\partial \mathbf{v}_w}{\partial t}\right]\!\right]\right). \tag{49}$$

Due to eqns (20) and (24) and to the incompressibility condition eqn (12), we have

$$\left[\!\left[\frac{\partial \mathbf{v}_w}{\partial t}\right]\!\right] = -\frac{n^s}{n^w}\left(\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right]\cdot\mathbf{n}\right)\mathbf{n}. \tag{50}$$

Similarly, due to the constitutive equation (2) and to the iterated geometrical condition of compatibility, eqn (39), $[\![\operatorname{div}\partial\mathbf{t}^w/\partial t]\!]$ is aligned with **n**. Also due to the incompressibility condition eqn (12) and to eqn (39), the induced discontinuities are balanced by the condition:

$$n^s(\boldsymbol{\Delta}_s \cdot \mathbf{n}) + n^w(\boldsymbol{\Delta}_w \cdot \mathbf{n}) = 0. \tag{51}$$

Therefore, the projection of eqn (49) along **n** can be written as

$$-2\rho^w \frac{\delta}{\delta t}\left(\frac{n^s}{n^w}\left(\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right]\cdot\mathbf{n}\right)\mathbf{n}\right) = \left[\!\left[\operatorname{div}\frac{\partial \mathbf{t}^w}{\partial t}\right]\!\right] + \rho^w W^2 \frac{n^s}{n^w}(\boldsymbol{\Delta}_s \cdot \mathbf{n})\mathbf{n} + \frac{\xi}{n^w}\left(\left[\!\left[\frac{\partial \mathbf{v}_s}{\partial t}\right]\!\right]\cdot\mathbf{n}\right)\mathbf{n}. \tag{52}$$

Subtraction of eqn (52) multiplied by n^s/n^w from eqn (48) yields the evolution equation for the 3-component vector $\mathbf{V} = [\![\partial \mathbf{v}_s/\partial t]\!]$ in the solid phase

$$2\mathbf{M}\cdot\frac{\delta \mathbf{V}}{\delta t} = (\mathbf{A}^{*ss} - W^2\mathbf{M})\cdot\boldsymbol{\Delta}_s - \xi\frac{\mathbf{E}}{\rho^s}\cdot\mathbf{V} \tag{53}$$

where the 3×3 matrix **M** has been defined by eqn (31) and the 3×3 symmetric matrix **E** takes a similar form:

$$\mathbf{E} = \rho^s\left(\boldsymbol{\delta} + \frac{n^s}{n^w}\mathbf{n}\otimes\mathbf{n}\right)^2 = \rho^s\left(\boldsymbol{\delta} + \left(\frac{1}{(n^w)^2} - 1\right)\mathbf{n}\otimes\mathbf{n}\right). \tag{54}$$

The wave-front in the fluid-phase is then given by eqn (50). Since the differential equation (53) has the same form as for compressible constituents, eqn (42), the evolution of the amplitude of the vector **V** is still given by eqn (44) but now with the matrices **M** and **E** given respectively by eqns (31) and (54) and with the 3-component vectors $\mathbf{e}^L$ and $\mathbf{e}^R$ now left and right eigenvectors of the generalized eigenvalue problem (30).

Remark 4.3

Bounds for the decay coefficient. For compressible constituents, the fact that **E**, eqn (43), is only semi-definite positive implies the possibility for the wave to propagate without diffusion, a situation termed *dynamically compatible*. This case has been analyzed for an elastic mixture by Biot (1956) and Bowen (1976). Their result will be rederived later using the procedure outlined in the Remark 4.1 above. Instead of using the eigenvectors associated to the generalized eigenvalue problem (22), one may use the left $\mathbf{e}^{\#L} = \mathbf{e}^{L\#}$ and right $\mathbf{e}^{\#R} = \mathbf{e}^{R\#}$ eigenvectors defined by the normalization eqn (26). Then the decay coefficient takes the form:

$$X = \frac{\mathbf{e}^{\#L}\cdot\mathbf{E}^{\#}\cdot\mathbf{e}^{\#R}}{\mathbf{e}^{\#L}\cdot\mathbf{e}^{\#R}}. \tag{55}$$

Thus for elastic or elastic–plastic constituents with an associative flow rule, the decay coefficient appears in the form of a Rayleigh quotient associated to the symmetric semi-definite positive matrix $\mathbf{E}^{\#}$. Consequently, it is bounded below and above by the smallest and largest eigenvalues of this matrix:

$$0 \leqslant X \leqslant 1 + \frac{\rho^s}{\rho^w} \quad \text{for elasticity or associative elastoplasticity.} \tag{56}$$

In the next section, these bounds will be tightened for incompressible constituents.

5. GROWTH AND DECAY FOR INCOMPRESSIBLE CONSTITUENTS

For a plastic eigenvalue W^2 of single multiplicity, the decay coefficient X can be calculated using the procedure described in Remark 4.1. One computes first the eigenvalues $W^2(\zeta)$ that make singular the matrix $\mathbf{A}^{*ss}+\zeta\mathbf{E}-W^2\mathbf{M}$ or equivalently the matrix $\mathbf{Z}$:

$$\mathbf{Z} = \frac{\mathbf{A}^{*ss}}{\rho^s} + \zeta\frac{\mathbf{E}}{\rho^s} - W^2\frac{\mathbf{M}}{\rho^s}. \tag{57}$$

The determinant of the latter can be cast in the form:

$$\det \mathbf{Z} = ((c_s^e)^2 + \zeta - W^2)F(W^2), \tag{58}$$

$$F(W^2) = u_0(W^2)^2 + u_1(\zeta)W^2 + u_2(\zeta). \tag{59}$$

The coefficients u_0 and $u_i(\zeta)$, $i = 1$ to 2, above are given in Appendix A. Notice first that, if the shear wave-speed c_s^e has multiplicity one, it follows from the factorization above and eqn (46) that the associated decay coefficient is equal to one like for an elastic behaviour (Biot, 1956).

Let us now turn to the study of the decay coefficients associated to the plastic wave-speeds. If the solutions $W^2_{\varepsilon_w}$, $\varepsilon_w = \pm 1$, to $F(W^2) = 0$ are real, namely

$$W^2 = \frac{1}{2u_0}(-u_1(\zeta) + \varepsilon_w\sqrt{\Delta(\zeta)}), \quad \Delta(\zeta) = u_1^2(\zeta) - 4u_0u_2(\zeta) \geqslant 0, \tag{60}$$

then the coefficients $X_{\varepsilon_w} = \mathrm{d}W^2_{\varepsilon_w}/\mathrm{d}\zeta(\zeta = 0)$ associated to $W^2_{\varepsilon_w}$ are simply:

$$X_{\varepsilon_w} = -\frac{1}{2u_0}\left(\frac{\mathrm{d}u_1}{\mathrm{d}\zeta} + \frac{\varepsilon_W}{\sqrt{\Delta}}\left(-u_1\frac{\mathrm{d}u_1}{\mathrm{d}\zeta} + 2u_0\frac{\mathrm{d}u_2}{\mathrm{d}\zeta}\right)\right) \quad (\zeta = 0). \tag{61}$$

If the roots to $F(W^2) = 0$ are complex conjugate, then the real part of X_{ε_w} is the same for both roots and

$$\mathrm{Re}(X_{\pm 1}) = -\frac{1}{2u_0}\frac{\mathrm{d}u_1}{\mathrm{d}\zeta}(\zeta = 0). \tag{62}$$

When applied to the coefficients u_i, $i = 0$ to 2, Appendix A, the above formulas become, outside the flutter region where the squares of the wave-speeds are real,

$$X_{\varepsilon_w} = \frac{1}{2}\left(1 + \frac{1}{r(n^w)^2}\right) - \frac{\varepsilon_W}{2\sqrt{\Delta(\zeta = 0)}}\left(1 - \frac{1}{r(n^w)^2}\right)\left(1 + \frac{2\mu_s}{H}\frac{x + ry}{\tau}\right)\tau, \tag{63}$$

while, in the flutter region where the squares of the wave-speeds are complex conjugate,

$$\mathrm{Re}(X_{\pm 1}) = \frac{1}{2}\left(1 + \frac{1}{r(n^w)^2}\right) > 0 \tag{64}$$

is independent of the plastic material constants.

Key parameters that decide of the nature of the wave-speeds are, besides the normalized plastic modulus $H/2\mu_s$, the scalar τ involving the longitudinal and shear elastic wave-speeds c_l^e and c_s^e, respectively,

$$\tau = r((c_l^e)^2 - (c_s^e)^2) \tag{65}$$

and the couple (ry, x), with r defined by eqn (32) and x and y by eqn (A.5). Indeed, the discriminant $\Delta(0)$, eqn (60), may be expressed in terms of these parameters:

$$\Delta = \left(\frac{2\mu_s}{H}\right)^2 (ry-x)^2 + 2\left(\frac{2\mu_s}{H}\right)(ry+x)\tau + \tau^2. \tag{66}$$

In the elastic limit, $H \to \infty$, the decay coefficients (63) simplify due to (66):

$$X_{\varepsilon_W} = \frac{1}{2}\left(1 + \frac{1}{r(n^{\text{w}})^2}\right) - \frac{\varepsilon}{2}\left(1 - \frac{1}{r(n^{\text{w}})^2}\right) = \begin{cases} 1 & \text{for } W = c_s^e \\ \dfrac{1}{r(n^{\text{w}})^2} & \text{for } W = c_l^e \end{cases} \tag{67}$$

where $\varepsilon = \varepsilon_W \text{sign}(\tau)$ is equal to -1 for the shear wave-speed and 1 for the longitudinal wave-speed. Notice that, due to the definition of r, eqn (32), the elastic longitudinal wave is always more diffusive than the shear wave, namely:

$$\frac{1}{r(n^{\text{w}})^2} > 1. \tag{68}$$

Also, notice that the decay coefficient in the flutter region is half way between the elastic decay coefficients.

The matrix **E**, eqn (54), is now symmetric positive-definite. Thus acceleration waves in incompressible constituents, whether elastic or elastic–plastic with an associative flow rule, can not propagate without diffusion, that is, the so-called dynamic compatibility is excluded. Moreover, the above inequality implies then that the decay coefficient X, eqn (55), is greater than one. Hence the lower and upper bounds, eqn (56), are strictly tightened to

$$1 \leqslant X \leqslant \frac{1}{r(n^{\text{w}})^2} \quad \text{for elasticity or associative elastoplasticity.} \tag{69}$$

These bounds do not hold for non-associative flow-rules. To elaborate further on that point, we assume deviatoric associativity to hold, that is only the deviatoric parts of the normals to the plastic potential **P** and to the yield surface **Q** are colinear. Then, the scalar $y = y(\mathbf{n})$, eqn (A.5), is positive or zero; since x is negative for an associative flow rule, a necessary but not sufficient condition for $x = x(\mathbf{n})$ to be positive is that the volumetric parts of **P** and **Q** are not equal. Moreover if τ is negative, complex wave-speeds are available for the directions **n** such that $x = x(\mathbf{n}) > 0$ and for normalized plastic moduli $H/2\mu_s$ within a certain range that renders the discriminant $\Delta(0)$, eqn (66), negative (see Loret and Harireche, 1991, for details).

A qualitative analysis of the shape and sign of the decay coefficients X associated to the two plastic wave-speeds involves also the normalized plastic modulus $H/2\mu_s$, the scalar τ and the directions of propagation **n** contained implicitly in the couple (ry, x). This analysis delineates four cases as follows.

Case a

$\tau < 0$, $x > 0$, $ry > 0$. Along the directions that ensure both x and ry to be strictly positive, flutter may occur for a certain range of plastic moduli. Indeed, the discriminant Δ is zero when the normalized plastic modulus is equal to $H_{\varepsilon_H}^{\text{flu}}/2\mu_s$, $\varepsilon_H = \pm 1$ where

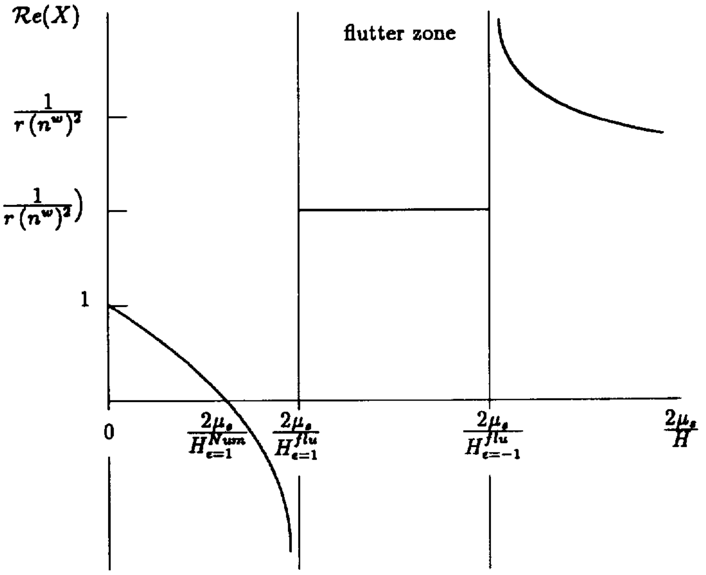

Fig. 1. Incompressible constituents. Sketch of the variations of the decay coefficient associated to the *largest* plastic wave-speed. The decay coefficient associated to the elastic shear-wave is 1 and the one associated to the elastic longitudinal wave is $1/r(n^w)^2$. Case a : $\tau < 0$, $x > 0$, $ry > 0$.

$$\frac{H^{\mathrm{flu}}_{\varepsilon_H}}{2\mu_s} = \left(\frac{1}{-\tau}\right)(\sqrt{ry} + \varepsilon_H\sqrt{x})^2 \tag{70}$$

and Δ is negative for $H \in [H^{\mathrm{flu}}_{\varepsilon=-1}, H^{\mathrm{flu}}_{\varepsilon=1}]$. Near the zeros of Δ, X_{ε_W} behaves essentially like $1/\sqrt{\Delta}$, to within a multiplicative coefficient whose sign is that of $-\varepsilon_H\varepsilon_W$.

The decay coefficient X_{ε_W} is zero when the normalized plastic modulus is equal to $H^{\mathrm{num}}_{\varepsilon=\varepsilon_W}/2\mu_s$ where

$$\frac{H^{\mathrm{num}}_{\varepsilon}}{2\mu_s} = \left(\frac{1}{-\tau}\right)\left(\sqrt{ry} + \varepsilon\frac{\sqrt{x}}{\sqrt{rn^w}}\right)(\sqrt{ry} + \varepsilon\sqrt{x}\sqrt{rn^w}) \tag{71}$$

should be positive, which is always true for the largest plastic wave-speed $\varepsilon_w = 1$; for the smallest plastic wave-speed $\varepsilon_W = -1$, the sign of this modulus depends on the propagation direction $\mathbf{n}$. Indeed, the following inequalities can be proved :

$$H^{\mathrm{num}}_{\varepsilon=1} \geq H^{\mathrm{flu}}_{\varepsilon=1} \geq H^{\mathrm{flu}}_{\varepsilon=-1} \geq \begin{cases} 0 \geq H^{\mathrm{num}}_{\varepsilon=-1} & \text{for } \sqrt{rn^w} < \dfrac{\sqrt{ry}}{\sqrt{x}} < \dfrac{1}{\sqrt{rn^w}}; \\ H^{\mathrm{num}}_{\varepsilon=-1} \geq 0 & \text{otherwise.} \end{cases} \tag{72}$$

The qualitative variations of the decay coefficients X as functions of the normalized modulus $H/2\mu_s$ are shown on Figs 1 and 2. Notice the behaviour of X at the extremities of the flutter region where the plastic wave-speed has multiplicity two : the eigenspace is defective by one dimension, the left and right eigenvectors are $\mathbf{M}$-orthogonal, eqn (47), so that $\mathbf{X}$ is unbounded.

Case b

$x < 0$, $ry > 0$. Now if x is negative, it is clear that none of the moduli that make zero the discriminant $\Delta(0)$ and the decay coefficients is real positive. Consequently, the decay coefficients remain real positive and finite (Fig. 3).

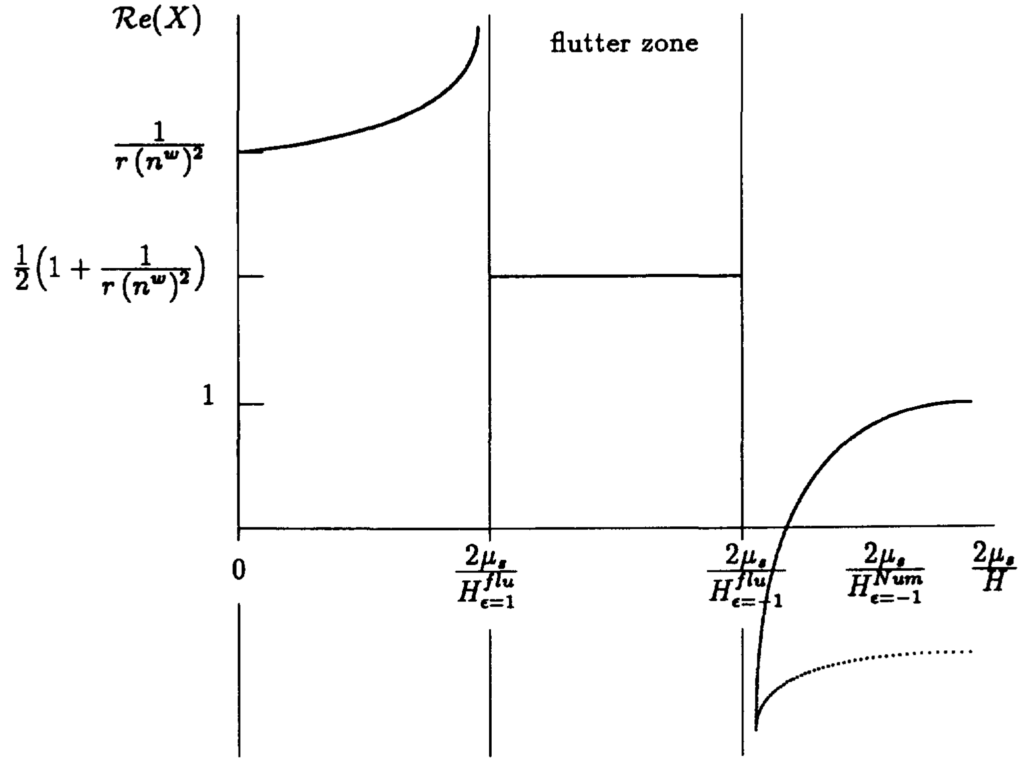

Fig. 2. Incompressible constituents. Sketch of the variations of the decay coefficient associated to the *smallest* plastic wave-speed: for small normalized moduli $H/2\mu_s$, the sign of the decay coefficient depends on the direction of propagation. Case a: $\tau < 0$, $x > 0$, $ry > 0$.

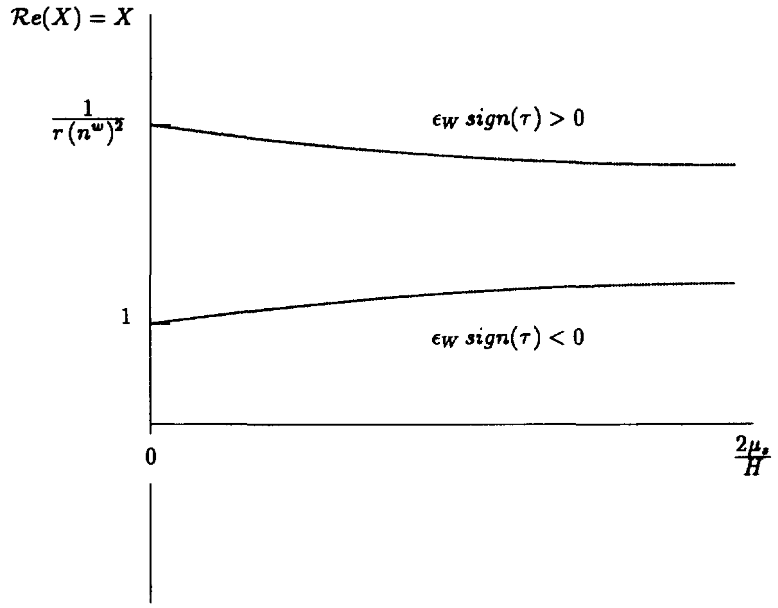

Fig. 3. Incompressible constituents. Sketch of the variations of the decay coefficients. This case is typical of an associative flow rule where $1 \leqslant X \leqslant 1/r(n^w)^2$. Case b: $x < 0$, $ry > 0$.

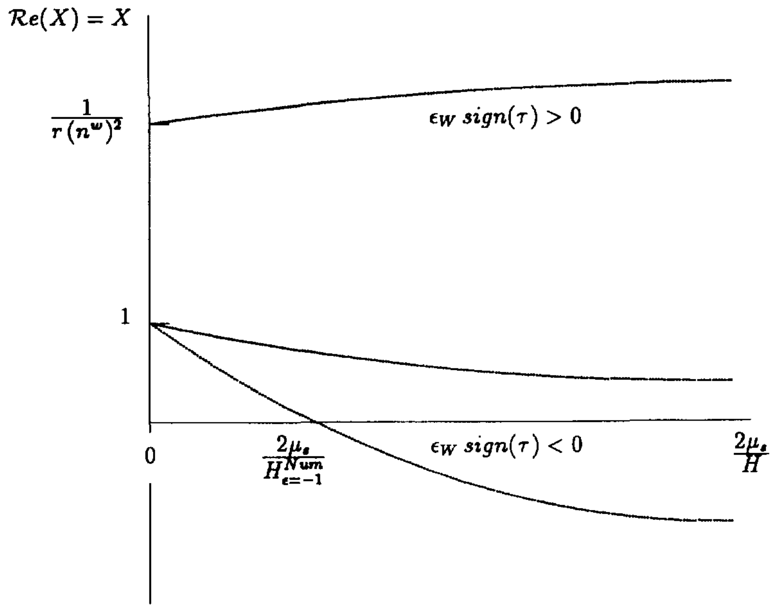

Fig. 4. Incompressible constituents. Sketch of the variations of the decay coefficients. The sign of the decay coefficient associated to the wave-speed which is equal to the shear wave-speed in the elastic limit depends on the direction of propagation. Case c: $\tau > 0$, $x > 0$, $ry > 0$.

Case c

$\tau > 0$, $x > 0$, $ry > 0$. The four moduli that make zero the discriminant $\Delta(0)$ and the decay coefficients can be ordered as follows:

$$\left.\begin{array}{lll} \text{for } \sqrt{r}n^w < \dfrac{\sqrt{ry}}{\sqrt{x}} < \dfrac{1}{\sqrt{r}n^w}, & & H^{\mathrm{num}}_{\varepsilon=-1} \geqslant 0 \\ \text{otherwise,} & & 0 \geqslant H^{\mathrm{num}}_{\varepsilon=-1} \end{array}\right\} \geqslant H^{\mathrm{flu}}_{\varepsilon=-1} \geqslant H^{\mathrm{flu}}_{\varepsilon=1} \geqslant H^{\mathrm{num}}_{\varepsilon=1}. \tag{73}$$

Consequently the discriminant $\Delta(0)$ is strictly positive so that the decay coefficients are finite. On the other hand, the coefficient $X_{\varepsilon_n=-1}$ associated to the smallest plastic wave-speed may be zero for some directions when $H^{\mathrm{num}}_{\varepsilon=-1}$ is positive (Fig. 4).

Case d

$x = 0$ or $ry = 0$. When x or ry are zero, at least one plastic wave-speed is equal to an elastic wave-speed, the so-called neutral wave: Table 1 summarizes the different possibilities and gives the associated eigenspaces. If the elastic shear-wave speed has multiplicity greater than one, the procedure defined in Remark 4.1 cannot be applied and the system of evolution equations (53) has to be considered directly. The analysis of all the particular cases given in Table 1 shows that the decay coefficient X and the part of the induced discontinuity Δ, which is orthogonal to the propagation vector $\mathbf{e}^R$ can be solved uniquely except when the eigenspaces are defective and of dimension 1; in the latter cases, additional equations would be needed to define the induced discontinuity.

Consequently, whether elastic or elastic–plastic with an associative flow rule, incompressible constituents are characterized by a strictly positive and bounded decay coefficient, eqn (69). In strong contrast, for non-associative flow rules, there may exist, depending on material parameters, directions along which acceleration waves grow exponentially in time, presumably giving rise to first-order waves. On the other hand, the decay coefficients associated to wave-speeds whose squares are complex are strictly positive.

Figures 5–9 are intended to quantify the qualitative features displayed by Figs 1–4. For that purpose, the material termed 'material 1' in Loret and Harireche (1991) is considered. Its characteristics are as follows:

Table 1. Right and left eigenvectors of the generalized eigenvalue problem for incompressible constituents. Notice that, in the elastic–plastic case **A**, the eigenspace is defective by one dimension when the two plastic wave-speeds coalesce

Wave speeds	Right eigenvectors	Left eigenvectors
	Elastic behaviour	
A.	if $\tau \neq 0$	idem
$W = c_l^e$ single root	$\mathbf{n}$	—
$W = c_s^e$ double root	plane $\perp \mathbf{n}$	—
B.	if $\tau = 0$	—
$W = c_s^e = c_l^e$ triple root	$\mathscr{R}^3$	—
	Elastic–plastic behaviour	
A.	if $x \neq 0,\ ry \neq 0$	
2 roots $W \neq c_s^e, \neq c_l^e$	$\mathbf{A}(\mathbf{a}, \mathbf{n})$	$\mathbf{A}(\mathbf{b}, \mathbf{n})$
$W = c_s^e$ single root	$\mathbf{n} \wedge \mathbf{b}$	$\mathbf{n} \wedge \mathbf{a}$
B.	if $x = 0,\ 2\mu_s ry + H\tau \neq 0,\ \tau \neq 0$	
1 root $W \neq c_s^e, \neq c_l^e$	$\mathbf{A}(\mathbf{a}, \mathbf{n})$	$\mathbf{A}(\mathbf{b}, \mathbf{n})$
$W = c_l^e$ single root	$\mathbf{B}(\mathbf{a}, \mathbf{b}, \mathbf{n})$	$\mathbf{B}(\mathbf{b}, \mathbf{a}, \mathbf{n})$
$W = c_s^e$ single root	$\mathbf{n} \wedge \mathbf{b}$	$\mathbf{n} \wedge \mathbf{a}$
C.	if $x = 0,\ 2\mu_s ry + H\tau = 0,\ \tau \neq 0$	
$W = c_l^e$ double root		
C1. $\mathbf{a} \cdot \mathbf{n} = 0,\ \mathbf{b} \cdot \mathbf{n} = 0$	plane $[\mathbf{a}, \mathbf{n}]$	plane $[\mathbf{b}, \mathbf{n}]$
C2. $\mathbf{a} \cdot \mathbf{n} \neq 0,\ \mathbf{b} \cdot \mathbf{n} = 0$	$\mathbf{n}$	$\mathbf{b}$
C3. $\mathbf{a} \cdot \mathbf{n} = 0,\ \mathbf{b} \cdot \mathbf{n} \neq 0$	$\mathbf{a}$	$\mathbf{n}$
$\mathbf{W} = \mathbf{c}_s^e$ single root	$\mathbf{n} \wedge \mathbf{b}$	$\mathbf{n} \wedge \mathbf{a}$
D.		
1 root $W \neq c_s^e, \neq c_l^e$	$\mathbf{A}(\mathbf{a}, \mathbf{n})$	$\mathbf{A}(\mathbf{b}, \mathbf{n})$
$W = c_s^e$ double root		
	if $ry = 0,\ 2\mu_s x + H\tau \neq 0,\ \tau \neq 0$ and	
D1. $\mathbf{a} \nsim \mathbf{n},\ \mathbf{b} \sim \mathbf{n}$	plane $\perp \mathbf{n}$	plane $[\mathbf{D}(\mathbf{a}, \mathbf{b}, \mathbf{n}), \mathbf{n} \wedge \mathbf{a}]$
D2. $\mathbf{a} \sim \mathbf{n},\ \mathbf{b} \nsim \mathbf{n}$	plane $[\mathbf{D}(\mathbf{b}, \mathbf{a}, \mathbf{n}), \mathbf{n} \wedge \mathbf{b}]$	plane $\perp \mathbf{n}$
D3. $\mathbf{a} \sim \mathbf{n},\ \mathbf{b} \sim \mathbf{n}$	plane $\perp \mathbf{n}$	plane $\perp \mathbf{n}$
D4. $\mathbf{a} \nsim \mathbf{n},\ \mathbf{b} \nsim \mathbf{n}$	$\mathbf{n} \wedge \mathbf{b}$	$\mathbf{n} \wedge \mathbf{a}$
	or if $\tau = 0,\ x - ry \neq 0$	
D5.	plane $\perp \mathbf{b}$	plane $\perp \mathbf{a}$
E. $W = c_s^e$ triple root		
	if $ry = 0,\ 2\mu_s x + H\tau = 0$ and	
E1. $\mathbf{a} \nsim \mathbf{n},\ \mathbf{b} \sim \mathbf{n}$	plane $\perp \mathbf{n}$	plane $[\mathbf{n}, \mathbf{n} \wedge \mathbf{a}]$
E2. $\mathbf{a} \sim \mathbf{n},\ \mathbf{b} \nsim \mathbf{n}$	plane $[\mathbf{n}, \mathbf{n} \wedge \mathbf{b}]$	plane $\perp \mathbf{n}$
E3. $\mathbf{a} \sim \mathbf{n},\ \mathbf{b} \sim \mathbf{n}$	$\mathscr{R}^3$	$\mathscr{R}^3$
E4. $\mathbf{a} \nsim \mathbf{n},\ \mathbf{b} \nsim \mathbf{n}$	$\mathbf{n} \wedge \mathbf{b}$	$\mathbf{n} \wedge \mathbf{a}$
	or if $\tau = 0,\ x - ry = 0$	
E5.	plane $\perp \mathbf{b}$	plane $\perp \mathbf{a}$

$\mathbf{A}(\mathbf{a}, \mathbf{n}) = (((c_s^e)^2 - W^2) - r((c_l^e)^2 - W^2))(\mathbf{a} \cdot \mathbf{n})\mathbf{n} + r((c_l^e)^2 - W^2)\mathbf{a}$

$\mathbf{B}(\mathbf{a}, \mathbf{b}, \mathbf{n}) = \rho^s(H\tau + 2\mu_s ry)\mathbf{n} - r(\mathbf{b} \cdot \mathbf{n})\mathbf{a}$

$\mathbf{D}(\mathbf{a}, \mathbf{b}, \mathbf{n}) = (\mathbf{b} \cdot \mathbf{n})|\mathbf{a} - (\mathbf{a} \cdot \mathbf{n})|^2\mathbf{n} + \rho^s(H\tau + 2\mu_s x)(\mathbf{a} - (\mathbf{a} \cdot \mathbf{n})\mathbf{n})$

$$n^w = 0.02, \quad \rho^s/\rho^w = 2.5n^s/n^w;$$

$$\mu_s = 6000\ \text{MPa}; \quad \lambda_s/\mu_s = 25(=> \tau > 0) \quad \text{or} \quad \lambda_s/\mu_s = 15(=> \tau < 0);$$

$$\text{friction angle } \psi = 30^\circ, \quad \text{dilatancy angle } \chi = 0^\circ, \quad \text{Lode angle } l = 20^\circ.$$

The plastic potential **P** and normal to the yield surface **Q** include the friction and dilatancy angles and the stress state involves the Lode angle l; these quantities are defined by eqns (3.32) and (5.5) in the above reference. Figures 5–9 display the decay coefficient X associated to the largest plastic wave-speed for appropriate ranges of the plastic moduli starting at the elastic limit, that is $2\mu_s/H = 0$. The directions of propagation **n** belong to the plane defined by the stress-eigenvectors associated to the major and minor eigenstresses, the angle made by the normals **n** and the direction of the major stress-eigenvector is denoted by θ; recall that the direction of propagation affects the sign of the scalar x, eqn (A5). When τ is negative, flutter is possible for a certain range of plastic moduli and in certain directions of propagation (see Figs 5 and 8(b) of Loret and Harireche, 1991). Figures 5–7 are in agreement with these observations. In contrast, when τ is positive, flutter is excluded for any plastic modulus and any direction of propagation as illustrated by Figs 8 and 9. The

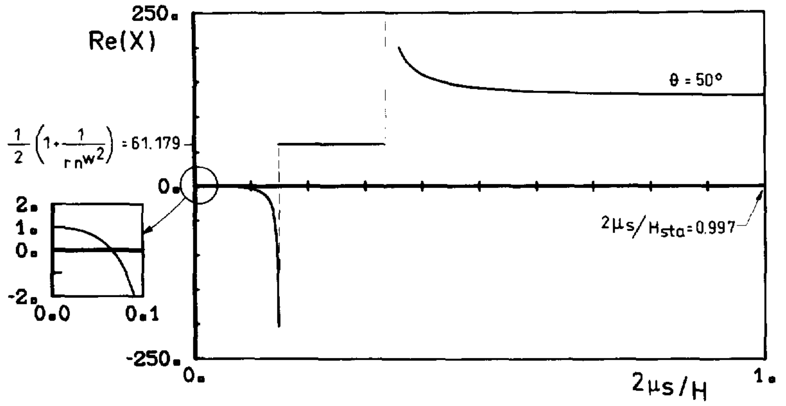

Fig. 5. Incompressible constituents, 'material 1' with $\lambda_s^*/\mu_s = 15$, leading to $\tau < 0$. Variations of the decay coefficient associated to the *largest* plastic wave-speed as a function of the plastic moduli. For the direction of propagation defined by $\theta = 50^\circ$, the scalar x is positive; consequently, this plot is a quantitative example of the sketch shown in Fig. 1.

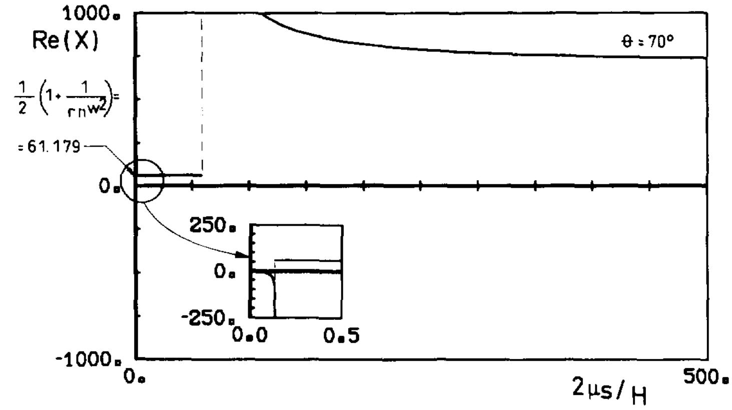

Fig. 6. Incompressible constituents, same as Fig. 5 but for a direction of propagation defined by $\theta = 70^\circ$ leading to $x > 0$. Notice the different ranges of interest with respect to Fig. 5 for the plastic moduli and decay coefficient.

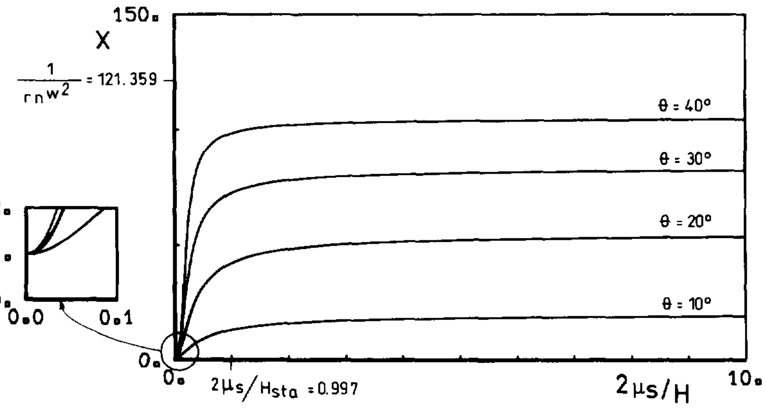

Fig. 7. Incompressible constituents, same as Fig. 5 but for directions of propagation leading to $x < 0$; consequently, this plot is a quantitative example of the lower curve shown in Fig. 3 with $\varepsilon_H = 1$ for the *largest* wave-speed.

important point here is to note the quantitative influence of the plastic moduli and of the directions of propagation on the values of the decay coefficients.

6. GROWTH AND DECAY FOR COMPRESSIBLE CONSTITUENTS

The analysis for compressible constituents parallels that of the previous section although the complexity of algebra prevents us to obtain completely explicit results. For a

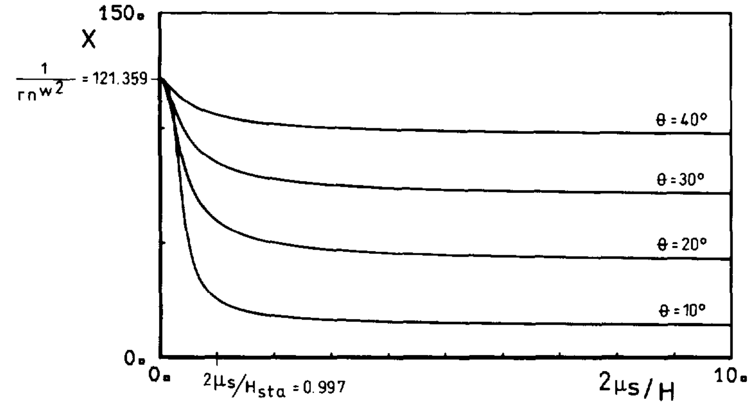

Fig. 8. Incompressible constituents, 'material 1' with $\lambda_s^*/\mu_s = 25$, leading to $\tau > 0$, thus excluding flutter and for directions of propagation associated to $x < 0$; consequently, this plot is a quantitative example of the upper curve shown in Fig. 3.

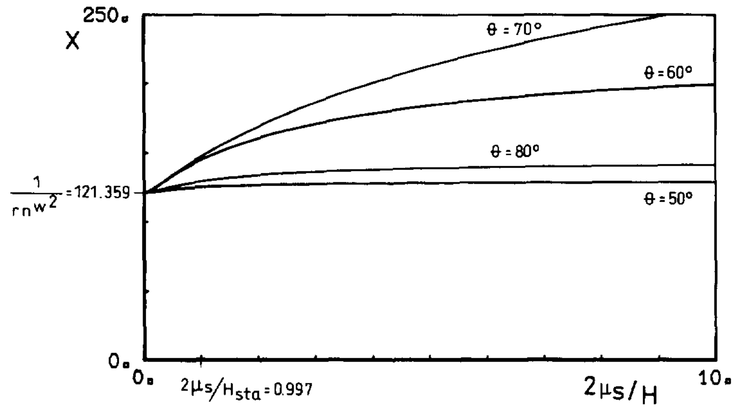

Fig. 9. Incompressible constituents, same as Fig. 8 but for directions of propagation leading to $x > 0$; consequently, this plot is a quantitative example of the upper curve shown in Fig. 4.

plastic eigenvalue W^2 of single multiplicity, one computes first the eigenvalues $W^2(\zeta)$ that make singular the matrix $\mathbf{Z}^\# = \mathbf{A}^\# + \zeta\mathbf{E}^\# - W^2\mathbf{I}_4$. The determinant of the matrix $\mathbf{Z}^\#$ can be cast in the form:

$$\det \mathbf{Z}^\# = -((c_s^e)^2 + \zeta - W^2)F(W^2), \tag{74}$$

$$F(W^2) = (W^2)^3 + u_1(\zeta)(W^2)^2 + u_2(\zeta)W^2 + u_3(\zeta). \tag{75}$$

The coefficients $u_i(\zeta)$, $i = 1$ to 3, above are given in Appendix B. Like for incompressible constituents, if the shear-wave speed has multiplicity one, one reads from eqn (74) that the associated decay coefficient is equal to 1. Let us consider now the plastic wave-speeds. In order to obtain the solutions W^2 of the cubic equation (75), let us introduce the coefficients:

$$p = u_2 - \frac{u_1^2}{3}, \quad q = \frac{2}{27}u_1^3 - \frac{1}{3}u_1u_2 + u_3, \tag{76}$$

$$R = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3, \quad u = \left(-\frac{q}{2} + \sqrt{R}\right)^{1/3}, \quad v = \left(-\frac{q}{2} - \sqrt{R}\right)^{1/3}. \tag{77}$$

The three roots are:

$$W_1^2 = -\frac{u_1}{3} + u + v. \tag{78}$$

$$W_2^2 = -\frac{u_1}{3} - \frac{u+v}{2} + i\frac{\sqrt{3}}{2}(u-v), \tag{79}$$

$$W_3^2 = -\frac{u_1}{3} - \frac{u+v}{2} - i\frac{\sqrt{3}}{2}(u-v). \tag{80}$$

If the discriminant R is negative, u and v are complex conjugate and the three roots are real. Thus the three decay coefficients are obtained by differentiation of eqn (75):

$$\frac{\mathrm{d}W_k^2}{\mathrm{d}\zeta} = -\frac{1}{3}\frac{\mathrm{d}u_1}{\mathrm{d}\zeta} - \frac{\dfrac{\mathrm{d}p}{\mathrm{d}\zeta}\left(W_k^2 + \dfrac{u_1}{3}\right) + \dfrac{\mathrm{d}q}{\mathrm{d}\zeta}}{3\left(W_k^2 + \dfrac{u_1}{3}\right)^2 + p}, \quad k \in [1, 3], \tag{81}$$

where the derivatives are estimated at $\zeta = 0$. Notice that the denominator appearing in the above expression is non-zero for the root W_1^2; on the other hand, it tends to zero as the two other roots tend to coalesce when the discriminant R approaches 0. Therefore, near a plastic double root, the decay coefficients are again unbounded.

If the discriminant R is positive, u and v are real and so W_1^2 is real and W_2^2 and W_3^2 are complex conjugate. Consequently in the directions and for the material parameters that imply R to be positive, the so-called flutter region, the decay coefficient associated to the real root W_1^2 is still given by eqn (81) for $k = 1$ and the decay coefficients associated to the complex conjugate roots are equal and bounded:

$$\mathrm{Re}\left(\frac{\mathrm{d}W_2^2}{\mathrm{d}\zeta}\right) = \mathrm{Re}\left(\frac{\mathrm{d}W_3^2}{\mathrm{d}\zeta}\right) = -\frac{1}{2}\frac{\mathrm{d}u_1}{\mathrm{d}\zeta} - \frac{1}{2}\frac{\mathrm{d}W_1^2}{\mathrm{d}\zeta}. \tag{82}$$

Remark 6.1

On dynamic compatibility. We have pointed out that the matrix $\mathbf{E}$, eqn (43), is semi-definite positive. Consequently, irrespective of the associative or non-associative character of the flow rule, there is the possibility of wave-propagation without diffusion. Let us see briefly how the elastic dynamic compatible case analyzed by Biot (1956) and Bowen (1976) is recovered using the procedure described in Remark 4.1. First, the elastic shear-wave speed has multiplicity at least two and so a direct method is required to calculate the associated decay coefficient. On the other hand, if the two longitudinal wave-speeds are distinct, say $c_{\mathrm{u}}^{\mathrm{e}} > c_l^{\mathrm{e}}$, eqn (B.19), the two remaining roots $W_{\varepsilon_H}^2(\zeta)$, $\varepsilon_H = \pm 1$, to eqn (75) are

$$\frac{1}{2}\left[\frac{\lambda_s + \rho^s\zeta + 2\mu_s}{\rho^s} + \frac{\lambda_w + \rho^s\zeta}{\rho^w}\right] + \frac{\varepsilon_H}{2}\left[\left(\frac{\lambda_s + \rho^s\zeta + 2\mu_s}{\rho^s} - \frac{\lambda_w + \rho^s\zeta}{\rho^w}\right)^2 + 4\frac{(\lambda_{sw} - \rho^s\zeta)^2}{\rho^s\rho^w}\right]^{1/2} \tag{83}$$

and therefore, at $\zeta = 0$,

$$X_{\varepsilon_H} = \frac{\mathrm{d}W_{\varepsilon_H}^2}{\mathrm{d}\zeta}(0) = \left(1 + \frac{\rho^s}{\rho^w}\right)\frac{(W_{\varepsilon_H}^2(0) - c_0^2)\varepsilon_H}{(c_{\mathrm{u}}^{\mathrm{e}})^2 - (c_l^{\mathrm{e}})^2} \geqslant 0, \tag{84}$$

where c_0 is termed the wave-speed of the 'frozen mixture':

$$(c_w^e)^2 \geqslant c_0^2 = \frac{\lambda_s + 2\mu_s + \lambda_w + 2\lambda_{sw}}{\rho^s + \rho^w} \geqslant (c_l^e)^2. \tag{85}$$

The speed c_0 is a longitudinal wave-speed if

$$\frac{\lambda_s + 2\mu_s + \lambda_{sw}}{\rho^s} = \frac{\lambda_w + \lambda_{sw}}{\rho^w} = c_0^2. \tag{86}$$

Notice that the second inequality in (85) is just a consequence of the first one. So if $((\lambda_w + \lambda_{sw})/\rho^w)^{1/2}$ is a longitudinal wave-speed, it is non-diffusive and the second longitudinal wave-speed, $(\lambda_w/\rho^w - \lambda_{sw}/\rho^s)^{1/2}$, has a decay coefficient $X = 1 + \rho^s/\rho^w$. Thus the upper and lower bounds defining the range of the decay coefficients, eqn (56), are realized simultaneously by the dynamically compatible parameters. If the constitutive parameter λ_{sw} is positive, like for sand, eqns (10), (11), the non-diffusive wave-speed is the largest one.

Figures 10–14 are intended to quantify the above discussion. For that purpose, the material termed 'material 4' in Loret and Harireche (1991) which is representative of a sand is considered. Its characteristics are as follows:

$$n^w = 0.19, \rho^s/\rho^w = 2.5n^s/n^w;$$

$$\mu_s = 6000 \text{ MPa}; \quad \lambda_s = 9183 \text{ MPa}; \quad \lambda_w = 493 \text{ MPa}; \quad \lambda_{sw} = 2102 \text{ MPa};$$

$$\text{friction angle } \psi = 15^\circ, \quad \text{dilatancy angle } \chi = 0^\circ, \quad \text{Lode angle } l = 20^\circ.$$

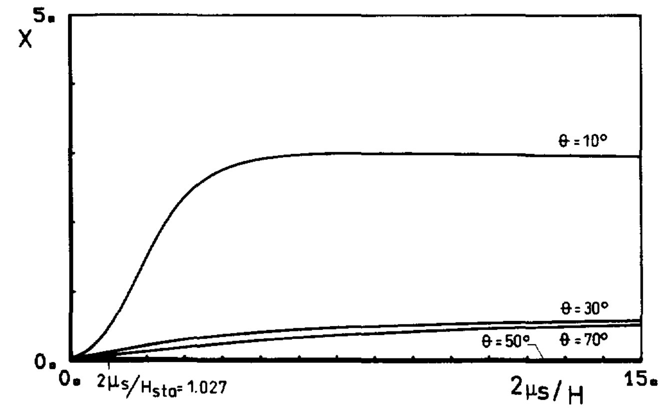

Fig. 10. Compressible constituents, 'material 4'. Variations of the decay coefficient associated to the *largest* plastic wave-speed as a function of the plastic moduli; this plot is reminiscent of the upper curve shown in Fig. 4, although the elastic values differ qualitatively as explained in the text.

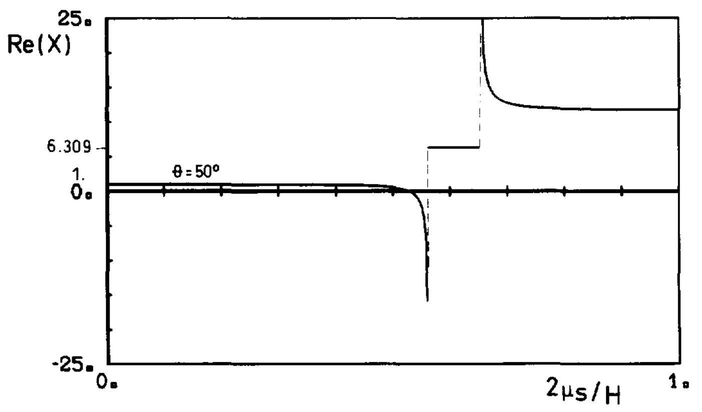

Fig. 11. Compressible constituents, same as Fig. 10 but for the *intermediate* plastic wave-speed in a direction for which flutter occurs in a certain range of plastic moduli; this plot is essentially similar to the sketch displayed in Fig. 1.

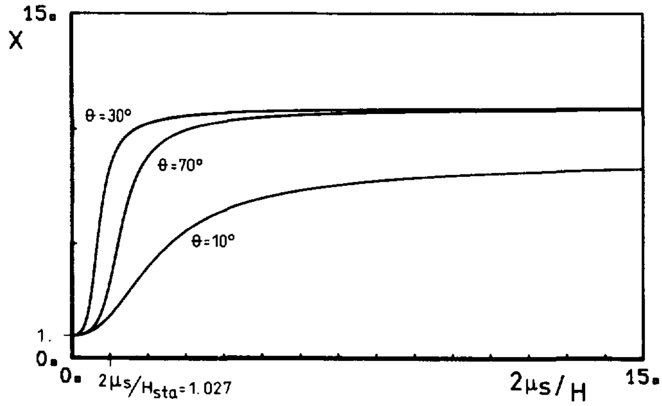

Fig. 12. Compressible constituents, same as Fig. 10 but for the *intermediate* plastic wave-speed in directions for which flutter is excluded; this plot is akin to the lower curve displayed in Fig. 3.

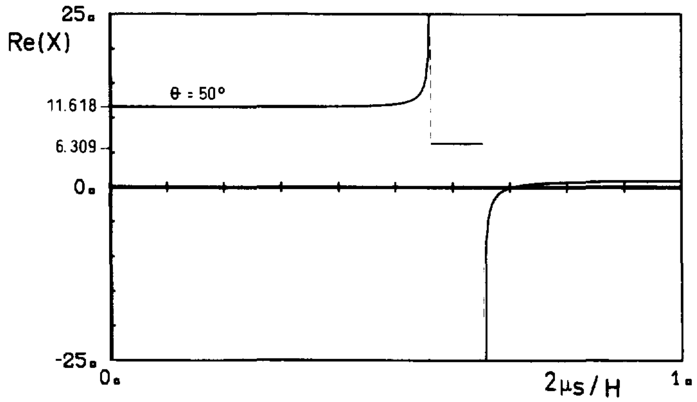

Fig. 13. Compressible constituents, same as Fig. 10 but for the *smallest* plastic wave-speed in a direction for which flutter occurs in a certain range of plastic moduli; this plot is akin to the sketch displayed in Fig. 2.

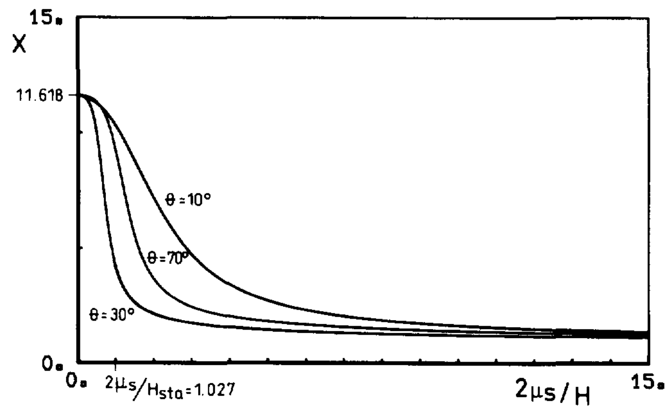

Fig. 14. Compressible constituents, same as Fig. 10 but for the *smallest* plastic wave-speed in directions for which flutter is excluded; this plot is akin to the upper curve displayed in Fig. 3.

It can be checked that this material is almost dynamically compatible in the elastic regime: indeed the elastic decay coefficient associated to the largest wave-speed is almost zero and it is quite close to $1+\rho^s/\rho^w$ for the second longitudinal wave (Figs 10 and 13/14, respectively). Figures 10–14 display the decay coefficient X associated to the largest, intermediate and smallest plastic wave-speeds for appropriate ranges of the plastic moduli starting at the elastic limit. As shown by Figs 11 and 13 of Loret and Harireche (1991), flutter occurs in a certain interval of plastic moduli and in a certain fan of directions of propagation; notice that the largest plastic wave-speed is always real while flutter is characterized by the squares of the intermediate and smallest plastic wave-speeds becoming complex conjugate. These features are useful to understand the variations of the decay coefficients shown in Figs 10–14. It may also be interesting to contrast the range of variations of these coefficients with those of the incompressible constituents shown in Figs 5–9.

7. CONCLUSIONS

For the elastic–plastic fluid-saturated porous media analyzed in this paper, an associative flow rule is a necessary and sufficient condition for the constitutive tensor moduli to display the major symmetry property. For associative flow rules, the decay coefficients of the plane acceleration waves are found to be positive and bounded by the decay coefficients in the elastic mixtures: the plane acceleration waves propagate with an amplitude that either strictly decreases or remains constant.

A special case of deviation with respect to associative flow rules has been examined here, the flow rules with deviatoric associativity. Analytic results have been obtained when both fluid and solid constituents are incompressible. It has been found that, depending on the material parameters, the decay coefficients associated to real plastic wave-speeds may be negative and/or unbounded: the plane acceleration waves that propagate in a certain fan of directions grow exponentially in time presumably giving rise to first-order waves. Inside the flutter region, that is when the squares of the plastic wave-speeds are complex conjugate, the decay coefficient is found to be strictly positive and the same for both plastic wave-speeds. However it is found that the decay coefficient becomes unbounded, with a positive or negative sign, as the two plastic wave-speeds tend to coalesce at the boundaries of the flutter region, the amplitude of the acceleration waves grows or decays unboundedly. When the fluid and the solid constituents are compressible, the results are found to have major qualitative features analogous to the incompressible case above.

These results concerning the flutter phenomena are qualitatively distinct from what would be obtained from an analysis in terms of harmonic waves propagating in hypoelastic materials (Rice, 1976): for such kind of analysis the amplitudes of the harmonic waves would grow in an oscillatory manner inside the flutter region.

Actually, the present analysis and results have features both similar to and distinct from what would be obtained with harmonic waves. Indeed, it can be shown that the wave-speeds provided by an harmonic analysis are equal, in the limit of an infinitely large pulsation, to the values obtained here. Also in both analyses, the viscous effects due to Darcy's law play no role on the determination of the onset of stationary discontinuities since such situation corresponds to the onset of (acceleration or harmonic) waves with zero speed of propagation. However in what concerns the onset of the flutter phenomenon, the viscous effects play no role in the acceleration wave analysis since there exists no velocity discontinuity at the wave front but they will be involved in the harmonic wave analysis since the viscous terms will be present in the corresponding governing momentum equations and characteristic equation.

Notice that the results presented in this study also differ qualitatively from those usually obtained in single phase elastic materials (e.g. Chen, 1973) where the amplitude of the acceleration waves may also grow but, when that happens, it becomes infinite within a finite time. On the other hand, we have exhibited an instance where both viscous damping and plastic dissipation may not preclude growth of the wave front. In that respect, it is worth mentioning that this phenomenon is also observed for materials with fading memory (Chen, 1973, p. 347) and for viscoelastic materials (e.g. D'Escatha, 1974). Also, it should be pointed out that the viscous (diffusion) coefficient ξ only enters the analysis as a multiplicative factor of the decay coefficient X so that it amplifies the growth or decay phenomenon but it is not involved in the sign of X.

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APPENDIX A: SPECTRAL ANALYSIS FOR INCOMPRESSIBLE CONSTITUENTS

A.1. Eigenvalues of the extended generalized eigenvalue problem

It is convenient to associate the matrix $\mathbf{Z}$,

$$\mathbf{Z} = \frac{\mathbf{A}^{*\mathrm{ss}}}{\rho^{\mathrm{s}}} + \zeta \frac{\mathbf{E}}{\rho^{\mathrm{s}}} - W^2 \frac{\mathbf{M}}{\rho^{\mathrm{s}}} \tag{A.1}$$

to the matrix $\mathbf{A}^{*\mathrm{ss}} + \zeta\mathbf{E} - W^2\mathbf{M}$ that defines the extended eigenvalue problem. The elastic contribution $\mathbf{Z}^{\mathrm{e}}$ to $\mathbf{Z}$ displays the elastic shear-wave speed c_s^{e} and the elastic longitudinal wave-speed c_l^{e}

$$c_s^{\mathrm{e}} = \sqrt{\frac{\mu_s}{\rho^{\mathrm{s}}}}, \quad c_l^{\mathrm{e}} = \sqrt{\frac{1}{r}\frac{\lambda_s + 2\mu_s}{\rho^{\mathrm{s}}}}, \tag{A.2}$$

which are ensured to be real by the constitutive inequalities eqns (10). The respective order of the shear and longitudinal wave-speeds depends on the material parameters λ_s, μ_s and r. If they are equal, any direction of space is an eigendirection (see Table 1 above). Recall that, for an elastic behaviour, the shear wave-speed has multiplicity at least two and the associated motion affects the solid only while the longitudinal wave affects both phases (Biot, 1956). Due to eqns (16) and (31),

$$\mathbf{Z}^{\mathrm{e}} = r(c_l^{\mathrm{e}})^2\mathbf{n}\otimes\mathbf{n} + (c_s^{\mathrm{e}})^2(\boldsymbol{\delta} - \mathbf{n}\otimes\mathbf{n}) - W^2(\boldsymbol{\delta} + (r-1)\mathbf{n}\otimes\mathbf{n}). \tag{A.3}$$

Plasticity contributes to $\mathbf{Z}$ by a non-symmetric dyadic product $-\mathbf{a}\otimes\mathbf{b}/H\rho^{\mathrm{s}}$ where $\mathbf{a}$ and $\mathbf{b}$ are the two vectors:

$$\mathbf{a} = (\mathbf{E}^{*\mathrm{s}} : \mathbf{P})\cdot\mathbf{n}, \quad \mathbf{b} = \mathbf{n}\cdot(\mathbf{E}^{*\mathrm{s}} : \mathbf{Q}). \tag{A.4}$$

To express $\mathbf{Z}$ in the axes $(\mathbf{e}_1 = \mathbf{n}, \mathbf{e}_2, \mathbf{e}_3)$, such that $\mathbf{b}\cdot\mathbf{e}_3 = 0$, it is instrumental to introduce the scalars x and y with dimension of square of wave-speed defined by

$$x = x(\mathbf{n}) = -\frac{(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})}{2\mu_s\rho^{\mathrm{s}}}, \quad y = y(\mathbf{n}) = \frac{\mathbf{a}\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})}{2\mu_s\rho^{\mathrm{s}}} \tag{A.5}$$

and then

$$\mathbf{Z} = \begin{bmatrix} r\left((c_f^e)^2 + \dfrac{\zeta}{r(n^w)^2} - W^2\right) + \dfrac{2\mu_s}{H} x & -\dfrac{1}{H}\dfrac{a_1 b_2}{\rho^s} & 0 \\ -\dfrac{1}{H}\dfrac{a_2 b_1}{\rho^s} & (c_s^e)^2 + \zeta - W^2 - \dfrac{2\mu_s}{H} y & 0 \\ -\dfrac{1}{H}\dfrac{a_3 b_1}{\rho^s} & -\dfrac{1}{H}\dfrac{a_3 b_2}{\rho^s} & (c_s^e)^2 + \zeta - W^2 \end{bmatrix}. \tag{A.6}$$

The determinant of the matrix $\mathbf{Z}$ can be cast in the form:

$$\det \mathbf{Z} = ((c_s^e)^2 + \zeta - W^2) F(W^2), \tag{A.7}$$

$$F(W^2) = u_0 (W^2)^2 + u_1(\zeta) W^2 + u_2(\zeta). \tag{A.8}$$

The coefficients u_i, $i = 0$ to 2, involve an elastic and an elastic–plastic contribution

$$u_i = u_{ie} + \frac{2\mu_s}{H} u_{ip}, \quad i = 0 \text{ to } 2. \tag{A.9}$$

For each $i \in [0, 2]$, the elastic contribution u_{ie} is a polynomial of degree i in ζ while the elastic–plastic contribution u_{ip} is a polynomial of degree max $(i-1, 0)$:

$$u_{0e}(\zeta) = r, \tag{A.10}$$

$$u_{0p}(\zeta) = 0, \tag{A.11}$$

$$u_{1e}(\zeta) = -r((c_s^e)^2 + (c_f^e)^2) - \zeta r\left(1 + \frac{1}{r(n^w)^2}\right), \tag{A.12}$$

$$u_{1p}(\zeta) = ry - x, \tag{A.13}$$

$$u_{2e}(\zeta) = r(c_s^e)^2 (c_f^e)^2 + \zeta\left(r(c_f^e)^2 + \frac{(c_s^e)^2}{(n^w)^2}\right) + \zeta^2 \frac{1}{(n^w)^2}, \tag{A.14}$$

$$u_{2p}(\zeta) = -ry(c_f^e)^2 + (c_s^e)^2 x + \zeta\left(x - \frac{y}{(n^w)^2}\right). \tag{A.15}$$

A.2. Eigenvectors of the generalized eigenvalue problem

It is convenient to highlight the elastic and plastic contributions to the matrix $\mathbf{Z} = \mathbf{Z}(\zeta = 0)$ associated to the generalized eigenvalue problem (30). Indeed, according to (A.3),

$$\mathbf{Z} = \mathbf{Z}^e - \frac{1}{H}\frac{\mathbf{a} \otimes \mathbf{b}}{\rho^s}. \tag{A.16}$$

Let W be a plastic wave-speed different from the elastic wave-speeds. Then the matrix $\mathbf{Z}^e$ is non-singular and the right eigenvector $\mathbf{e}^R$ is simply

$$\mathbf{e}^R = \frac{\mathbf{b} \cdot \mathbf{e}^R}{H\rho^s} (\mathbf{Z}^e)^{-1} \cdot \mathbf{a}. \tag{A.17}$$

Explicitly,

$$(\mathbf{Z}^e)^{-1} = \frac{1}{(c_s^e)^2 - W^2} \boldsymbol{\delta} + \left(\frac{1}{r((c_f^e)^2 - W^2)} - \frac{1}{(c_s^e)^2 - W^2}\right) \mathbf{n} \otimes \mathbf{n}. \tag{A.18}$$

For neutral waves, the strain-rate discontinuity is tangent to the yield surface in strain-space so that $\mathbf{b} \cdot \mathbf{e}^R$ is zero. Then either x or y is zero. At least one plastic wave-speed equals an elastic wave-speed and the eigenvectors $\mathbf{e}^R$ may describe a space of dimension one, two or three; alternatively the eigenspace may be defective by one or two dimensions. Table 1 summarizes all possibilities. Recall that these 3-component eigenvectors $\mathbf{e}^R$ describe the motion in the solid phase, the associated motion in the fluid phase is given by eqn (50), namely by $-n^s/n^w(\mathbf{e}^R \cdot \mathbf{n})\mathbf{n}$. The left eigenvectors $\mathbf{e}^L$ are simply obtained by exchanging the roles of the vectors $\mathbf{a}$ and $\mathbf{b}$ (see eqn A.16).

APPENDIX B: SPECTRAL ANALYSIS FOR COMPRESSIBLE CONSTITUENTS

B.1. Eigenvalues of the extended eigenvalue problem

For compressible constituents, it is more convenient to work on the eigenvalue problem of eqn (27) in order to highlight symmetries. Thus we define the extended eigenvalue problem by the matrix $\mathbf{Z}^\#$:

$$\mathbf{Z}^{\#} = \mathbf{A}^{\#} + \zeta \mathbf{E}^{\#} - W^2 \mathbf{I}_4. \tag{B.1}$$

Plasticity contributes to $\mathbf{Z}^{\#}$ by a non-symmetric dyadic product $-\mathbf{a}^{\#} \otimes \mathbf{b}^{\#}/H$ where $\mathbf{a}$ and $\mathbf{b}$ are the two constitutive vectors

$$\mathbf{a} = (\mathbf{E}^{\prime s} : \mathbf{P}) \cdot \mathbf{n}, \quad \mathbf{b} = \mathbf{n} \cdot (\mathbf{E}^{\prime s} : \mathbf{Q}) \tag{B.2}$$

but $\mathbf{a}$ and $\mathbf{b}$ follow transformations different from those of eigenvectors, eqn (26):

$$\mathbf{a}^{\#} = \mathbf{M}^{-1/2} \cdot \mathbf{a}, \quad \mathbf{b}^{\#} = \mathbf{M}^{-1/2} \cdot \mathbf{b}. \tag{B.3}$$

The determinant of the matrix $\mathbf{Z}^{\#}$ can be cast in the form:

$$\det \mathbf{Z}^{\#} = -((c_s^{\mathrm{e}})^2 + \zeta - W^2) F(W^2), \tag{B.4}$$

$$F(W^2) = (W^2)^3 + u_1(\zeta)(W^2)^2 + u_2(\zeta) W^2 + u_3(\zeta). \tag{B.5}$$

The coefficients u_i involve an elastic and an elastic–plastic contribution

$$u_i = u_{i\mathrm{e}} + \frac{1}{H} u_{i\mathrm{p}}, \quad i = 1 \text{ to } 3. \tag{B.6}$$

For each $i \in [1, 3]$, the elastic contribution $u_{i\mathrm{e}}$ is a polynomial of degree i in ζ while the elastic–plastic contribution $u_{i\mathrm{p}}$ is a polynomial of degree $i-1$ in ζ. The zeroth-order terms have been given in Loret and Harireche (1991) but they are recorded here for completeness:

$$u_{1\mathrm{e}}(0) = -((c_s^{\mathrm{e}})^2 + (c_l^{\mathrm{e}})^2 + (c_w^{\mathrm{e}})^2), \tag{B.7}$$

$$u_{2\mathrm{e}}(0) = (c_s^{\mathrm{e}} c_l^{\mathrm{e}})^2 + (c_l^{\mathrm{e}} c_w^{\mathrm{e}})^2 + (c_w^{\mathrm{e}} c_s^{\mathrm{e}})^2, \tag{B.8}$$

$$u_{3\mathrm{e}}(0) = -(c_s^{\mathrm{e}} c_l^{\mathrm{e}} c_w^{\mathrm{e}})^2. \tag{B.9}$$

$$u_{1\mathrm{p}}(0) = \frac{\mathbf{a} \cdot \mathbf{b}}{\rho^s} + \frac{\lambda_{sw}^2}{\rho^w} \operatorname{tr} \mathbf{P} \operatorname{tr} \mathbf{Q}. \tag{B.10}$$

$$u_{2\mathrm{p}}(0) = -\left(\frac{\lambda_s + 2\mu_s}{\rho^s} + \frac{\lambda_w}{\rho^w}\right)\frac{\mathbf{a} \cdot \mathbf{b}}{\rho^s} + \frac{\lambda_s + \mu_s}{\rho^s} \frac{(\mathbf{a} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n})}{\rho^s}$$
$$- \frac{\lambda_s + 3\mu_s}{\rho^s} \frac{\lambda_{sw}^2}{\rho^w} \operatorname{tr} \mathbf{P} \operatorname{tr} \mathbf{Q} + \frac{\lambda_{sw} \lambda_{sw}}{\rho^s \rho^w} (\operatorname{tr} \mathbf{Q} (\mathbf{a} \cdot \mathbf{n}) + \operatorname{tr} \mathbf{P} (\mathbf{b} \cdot \mathbf{n})), \tag{B.11}$$

$$u_{3\mathrm{p}}(0) = \left(\frac{\lambda_s^* + 2\mu_s}{\rho^s} \frac{\mathbf{a} \cdot \mathbf{b}}{\rho^s} - \frac{\lambda_s + \mu_s}{\rho^s} \frac{(\mathbf{a} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n})}{\rho^s}\right) \frac{\lambda_w}{\rho^w}$$
$$+ \left(\frac{\lambda_s + 2\mu_s}{\rho^s} \frac{\lambda_{sw}^2}{\rho^w} \operatorname{tr} \mathbf{P} \operatorname{tr} \mathbf{Q} - \frac{\lambda_{sw} \lambda_{sw}}{\rho^s \rho^w} (\operatorname{tr} \mathbf{Q} (\mathbf{a} \cdot \mathbf{n}) + \operatorname{tr} \mathbf{P} (\mathbf{b} \cdot \mathbf{n})\right) \frac{\mu_s}{\rho^s}. \tag{B.12}$$

The derivatives of interest are given below:

$$\frac{\mathrm{d}u_{1\mathrm{e}}}{\mathrm{d}\zeta}(0) = -\left(2 + \frac{\rho^s}{\rho^w}\right), \tag{B.13}$$

$$\frac{\mathrm{d}u_{2\mathrm{e}}}{\mathrm{d}\zeta}(0) = \frac{\lambda_s + 2\mu_s}{\rho^s} + \frac{\lambda_s + 2\mu_s + 2\lambda_w + 2\lambda_{sw}}{\rho^w}, \tag{B.14}$$

$$\frac{\mathrm{d}u_{3\mathrm{e}}}{\mathrm{d}\zeta}(0) = -\frac{\lambda_s}{\rho^s} \frac{\lambda_w}{\rho^w} - \frac{\mu_s}{\rho^s} \frac{\lambda_s + 3\lambda_w + 2\lambda_{sw} + 2\mu_s}{\rho^w}, \tag{B.15}$$

$$\frac{\mathrm{d}u_{1\mathrm{p}}}{\mathrm{d}\zeta}(0) = 0, \tag{B.16}$$

$$\frac{\mathrm{d}u_{2\mathrm{p}}}{\mathrm{d}\zeta}(0) = -\left(1 + \frac{\rho^s}{\rho^w}\right)\frac{\mathbf{a} \cdot \mathbf{b}}{\rho^s} - 2\frac{\lambda_{sw}^2}{\rho^w} \operatorname{tr} \mathbf{P} \operatorname{tr} \mathbf{Q} - \frac{\lambda_{sw}}{\rho^w} (\operatorname{tr} \mathbf{Q} (\mathbf{a} \cdot \mathbf{n}) + \operatorname{tr} \mathbf{P} (\mathbf{b} \cdot \mathbf{n})), \tag{B.17}$$

$$\frac{\mathrm{d}u_{3p}}{\mathrm{d}\zeta}(0) = \frac{\lambda_s - 2\mu_s + \lambda_w + 2\lambda_{sw}}{\rho^w}\frac{\mathbf{a}\cdot\mathbf{b}}{\rho^s} - \frac{\lambda_s + \mu_s + 2\lambda_{sw}}{\rho^w}\frac{(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})}{\rho^s}$$

$$+\frac{\lambda_s + 3\mu_s}{\rho^s}\frac{\lambda_{sw}^2}{\rho^w}\,\mathrm{tr}\,\mathbf{P}\,\mathrm{tr}\,\mathbf{Q} - \frac{\lambda_{sw}(\lambda_{sw} - \mu_s)}{\rho^s\rho^w}(\mathrm{tr}\,\mathbf{Q}(\mathbf{a}\cdot\mathbf{n}) + \mathrm{tr}\,\mathbf{P}(\mathbf{b}\cdot\mathbf{n})). \quad \text{(B.18)}$$

The above coefficients use the three elastic wave-speeds, namely the shear wave-speed c_s^e and the two longitudinal wave-speeds c_l^e and c_w^e. The latter are defined implicitly by the following relations:

$$(c_s^e)^2 = \frac{\mu_s}{\rho^s}, \quad (c_l^e)^2 + (c_w^e)^2 = \frac{\lambda_s + 2\mu_s}{\rho^s} + \frac{\lambda_w}{\rho^w}, \quad (c_l^e)^2(c_w^e)^2 = \frac{\lambda_s + 2\mu_s}{\rho^s}\frac{\lambda_w}{\rho^w}. \quad \text{(B.19)}$$

B.2. Eigenvectors of the normalized eigenvalue problem

It is convenient to highlight the elastic and plastic contributions to the matrix $\mathbf{Z}^{\#}(\zeta = 0)$ associated to the normalized eigenvalue problem (27). Indeed,

$$\mathbf{Z}^{\#} = \mathbf{Z}^{\#e} - \frac{1}{H}\mathbf{a}^{\#}\otimes\mathbf{b}^{\#}. \quad \text{(B.20)}$$

The elastic contribution $\mathbf{Z}^{\#e}$ to $\mathbf{Z}^{\#}$ may be expressed componentwise using cartesian coordinates $(\mathbf{e}_1 = \mathbf{n}, \mathbf{e}_2, \mathbf{e}_3)$:

$$\mathbf{Z}^{\#e} = \begin{bmatrix} \dfrac{\lambda_s + 2\mu_s}{\rho^s} - W^2 & 0 & 0 & \dfrac{\lambda_{sw}}{\sqrt{\rho^s\rho^w}} \\ 0 & \dfrac{\mu_s}{\rho^s} - W^2 & 0 & 0 \\ 0 & 0 & \dfrac{\mu_s}{\rho^s} - W^2 & 0 \\ \dfrac{\lambda_{sw}}{\sqrt{\rho^s\rho^w}} & 0 & 0 & \dfrac{\lambda_w}{\rho^w} - W^2 \end{bmatrix}. \quad \text{(B.21)}$$

Let W be a plastic wave-speed different from the elastic wave-speeds. Then the matrix $\mathbf{Z}^{\#e}$ is non-singular and the right eigenvector $\mathbf{e}^{\#R}$ is simply

$$\mathbf{e}^{\#R} = \frac{\mathbf{b}^{\#R}\cdot\mathbf{e}^{\#R}}{H}(\mathbf{Z}^{\#e})^{-1}\cdot\mathbf{a}^{\#} \quad \text{(B.22)}$$

or componentwise in terms of the original vector $\mathbf{a}$:

$$\mathbf{e}^{\#R} = \frac{\mathbf{b}^{\#}\cdot\mathbf{e}^{\#R}}{H}\begin{bmatrix} ((\lambda_w/\rho^w - W^2)a_1 - (\lambda_{sw}\lambda_{sw}/\rho^w)\,\mathrm{tr}\,\mathbf{P})/(\sqrt{\rho^s}f_{lw}) \\ a_2/(\sqrt{\rho^s}f_s) \\ a_3/(\sqrt{\rho^s}f_s) \\ (-(\lambda_{sw}/\rho^s)a_1 + ((\lambda_s + 2\mu_s)/\rho^s - W^2)\lambda_{sw}\,\mathrm{tr}\,\mathbf{P})/(\sqrt{\rho^w}f_{lw}) \end{bmatrix}. \quad \text{(B.23)}$$

where

$$f_{lw} = ((c_l^e)^2 - W^2)((c_w^e)^2 - W^2), \quad f_s = (c_s^e)^2 - W^2. \quad \text{(B.24)}$$

For neutral waves, the strain-rate discontinuity is tangent to the yield surface in strain-space so that $\mathbf{b}^{\#}\cdot\mathbf{e}^{\#R} = \mathbf{b}\cdot\mathbf{e}^{R}$ is zero and at least one plastic wave-speed equals an elastic wave-speed. The complete analysis is not detailed here. Recall that the three first components of these 4-component eigenvectors $\mathbf{e}^{\#R}$ or $\mathbf{e}^{R}$ describe the motion in the solid phase, the associated motion in the fluid phase is given by the fourth component. The left eigenvectors $\mathbf{e}^{\#L}$ are simply obtained by exchanging the roles of the vectors $\mathbf{a}^{\#}$ and $\mathbf{b}^{\#}$ (see eqn B.20).